Statistical Machine Translation
Probability Theory II

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Recap

Probability space

A probability space is a triple $(\Omega, \mathfrak{A}, P)$, where:

- Ω is the sample space (the set of possible outcomes)
- $\mathfrak{A} \subseteq \wp(\Omega)$ is the algebra of events
- $P : \mathfrak{A} \to [0, 1]$ is the function assigning probabilities to events

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Today

- Random variables
- Independence
- Bayes' theorem

Partition

Special case

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$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i) \tag{5}$$

Throwing coins

Proposition

Suppose you throw a coin n times and that the probability of getting *heads* is h. Then, the probability of throwing heads k times is:

$$p(k) = \binom{n}{k} \times h^k \times (1 - h)^{(n-k)} \tag{6}$$

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Interpretation

- The probability of a sequence with k heads and $n-k$ tails is $h^k \times (1-h)^{(n-k)}$
- $\binom{n}{k}$ is the number of distinct sequences with k heads and $n-k$ tails

Random Variables

What we assume

Let $(\Omega, \mathfrak{A}, P)$ be a probability space. For simplicity, we assume that it is *discrete*:

- $\mathfrak{A} = \wp(\Omega)$ (all events and outcomes are possible)
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- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Random variable:

$$X(\omega) = \begin{cases} 60 & \text{if } \omega = 6 \\ -10 & \text{otherwise.} \end{cases}$$

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The Law of Large Numbers

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In our casino example, if the dice is fair, the player will win, in the long run, $1\frac{2}{3}$\$ per roll.

Or, if $p(6) = \frac{1}{8}$, loose 1.25\$ per roll.

Variance and Standard Deviation

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The *variance* of X measures the extent to which the actual values of the variable differ from the expected one:

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Intuitively, the expected gain of the player is therefore equal to $1\frac{2}{3} \pm 26$.

Example

Setup

Let's change the rules of the game:

- As before: if you roll 6, you win 60$; otherwise, you lose 10$.
- The change: you have to roll the die 5 times, no more, no less.

Questions

- What is the expected gain of the player in this new version of the game?
- What is the variance and standard deviation of the gain?

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- if α is a constant, then $\mathbb{E}(\alpha) = \alpha$
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- $\mathbb{V}(X + Y) = \mathbb{V}(Y) + \mathbb{V}(Y)$, **provided that X and Y are independent.**

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Proof (intuition)

- $\{Y = y : y \in \mathrm{Img}(Y)\}$ is a partition of Ω
- Let $A \equiv (X = x)$ and $B_y \equiv (Y = y)$. Then, Eq. 15 follows directly from Eq. 5.

Independence

Definition

We say that two events $A, B \in \mathfrak{A}$ are **independent** if the following holds:

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Definition

Let X, Y be two random variables. We say that X and Y are independent if for each possible value x of X and each possible value y of Y it holds that:

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y) \tag{18}$$

Conditional Independence

Definition

Let $A, B, C \in \mathfrak{A}$ be three events. We say that A and B are **(conditionally) independent** given C if:

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Intuitively, knowing B does not tell us anything about A if we already know C. And vice versa, knowing A does not tell us anything about B if we already know C.

Question

Let's $A, B, C \in \mathfrak{A}$ be three events. Suppose we don't know anything about them. Which of the following two assumptions is stronger?

- A and B are independent
- A and B are independent given C

Conditional Independence

Example

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Which of the following can be simplified/reduced and how?

- $P(C, U)$
- $P(C, R|U)$
- $P(C, U|R)$

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- R – it is raining
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Questions

Which of the following can be simplified/reduced and how?

- $P(C, U)$
- $P(C, R|U)$
- $P(C, U|R)$

You should adopt certain rational assumptions:

- Hans does not take umbrella on his way to work every day
- Hans does not use his umbrella to break the headlights of the cars passing by
- ...

Bayes' theorem

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

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Bayes' theorem: example

Example

Suppose we know that we have a biased coin, with $h = 0.4$. We throw the coin and get the following sequence:

$$H, T, T, T, H, H, T, H, T, H \tag{22}$$

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We can calculate the probability of such an event happening:

$$p(5) = \binom{10}{5} \times 0.4^5 \times 0.6^5 = 0.201$$

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Question

Let's assume that we know that the coin is either biased with $h = 0.4$ or not biased at all ($h = 0.5$). What is the probability of the coin being biased if we throw 5 heads out of 10?

Bayes' theorem: example

Events

- B – the coin is biased with $h = 0.4$
- N – the coin is not biased ($h = 0.5$)
- E – we get heads 5 times in 10 trials

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Let $\alpha := P(B)$:

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$$P(B|E) =$$

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Let $\alpha := P(B)$:

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$$0.201 \cdot \alpha + 0.236 \cdot (1 - \alpha) =$$

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$$0.201 \cdot \alpha + 0.236 \cdot (1 - \alpha) = 0.236 - 0.035\alpha$$

Bayes' theorem: example

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Result

$$P(B|E) = 0.201 \cdot \frac{\alpha}{0.236 - 0.035\alpha}$$

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$P(B) = \alpha$ can be seen as a **parameter** representing our **prior** knowlege about the coin.

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- if $\alpha = 0.5$, then $P(B|E) = 0.46$

Bayes' theorem: example

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Bayes' theorem: example

Events

- B – the coin is biased with $h = 0.4$
- N – the coin is not biased ($h = 0.5$)
- E – we get heads 5 time

Result

$$P(B|E) = 0.201 \cdot \frac{\alpha}{0.236 - 0.035\alpha}$$

Prior

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Estimation

- *Maximum likelihood esimates* (MLE):

$$\arg\max_{\alpha} P(D|\alpha) \tag{24}$$

- *Maximum a-posteriori esimates* (MAP):

$$\arg\max_{\alpha} P(\alpha|D) \tag{25}$$