How formal concept lattices solve a problem of ancient linguistics

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Pāṇini's Śivasūtras

अइउण्॥ऋऌक्॥एओङ्॥ऐऔच्॥हयवरट्॥लण्॥ञमङणनम्॥झभञ्।
।घढधष्॥जबगडदश्॥खफछठथचटतव्॥कपय्॥शषसर्॥हल्॥

a·i·uṇ||ṛ·ḷk||e·oṅ||ai·auc||hayavaraṭ|| laṇ||ñamaṅaṇanam||jhabhañ||
ghaḍhadhaṣ||jabagaḍadaś||khaphachaṭhathacaṭatav||kapay||śaṣasar||hal||

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A is replaced by **B** if preceded by **C** and followed by **D**

- in modern form: $A \rightarrow B/_{C\_D}$
- as context-sensitive rule: $CAD \rightarrow CBD$

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$$\begin{bmatrix} +consonantal \\ -nasal \\ +voiced \end{bmatrix} \rightarrow \begin{bmatrix} +consonantal \\ -nasal \\ -voiced \end{bmatrix} /\_\#$$

Pāṇini's coding of rules

$$A \to B /_{C\_\_D}$$

Pāṇini's coding of rules

$$A \rightarrow B/_{C\_\_D}$$

A + genitive, B + nominative, C + ablative, d + locative

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$$[\text{iK}] \rightarrow [\text{y}\d{N}]/\_\_[\text{aC}]$$

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Pāṇini's Śivasūtras

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Pāṇini's Śivasūtras

sūtras						anubandha
1.	a	i	u			Ṇ
2.				ṛ	ḷ	K
3.		e	o			Ṅ
4.		ai	au			C
5.	h	y	v	r		Ṭ
6.					l	Ṇ
7.	ñ	m	ṅ	ṇ	n	M
8.	jh	bh				Ñ
9.			gh	ḍh	dh	Ṣ
10.	j	b	g	ḍ	d	Ś
11.	kh	ph	ch	ṭh	th	
			c	ṭ	t	V
12.	k	p				Y
13.		ś	ṣ	s		R
14.	h					L

Pāṇini's Śivasūtras

anubandha

sūtras

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			c	ṭ	t	V
12.	k	p				Y
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14.	h					L

Phonological classes/ *pratyāhāras*

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8.	jh	bh				Ñ

Phonological classes are denoted by *pratyāhāras.*
E.g., the *pratyāhāra iC* denotes the set of segments in the continuous sequence starting with *i* and ending with *au*, the last element before the *anubandha C.*

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Minimality criteria

1. The length of the whole list is minimal.
2. The length of the sublist of the anubandhas is minimal and the length of the whole list is as short as possible.
3. The length of the sublist of the sounds is minimal and the length of the whole list is as short as possible.

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3. The length of the sublist of the sounds is minimal and the length of the whole list is as short as possible.
   - no duplication of *h*
   - less anubandhas

Basic concepts

S-encodable set of sets: $\Phi=\{\{d,e\},\{b,c,d,f,g,h,i\},\{a,b\},\{f,i\},\{c,d,e,f,g,h,i\},\{g,h\}\}$

S-alphabet $(\mathcal{A},\Sigma,<)$ of Φ: e d M_1 c i f M_2 g h M_3 b M_4 a M_5

alphabet

marker

total order on $\mathcal{A}\cup\Sigma$

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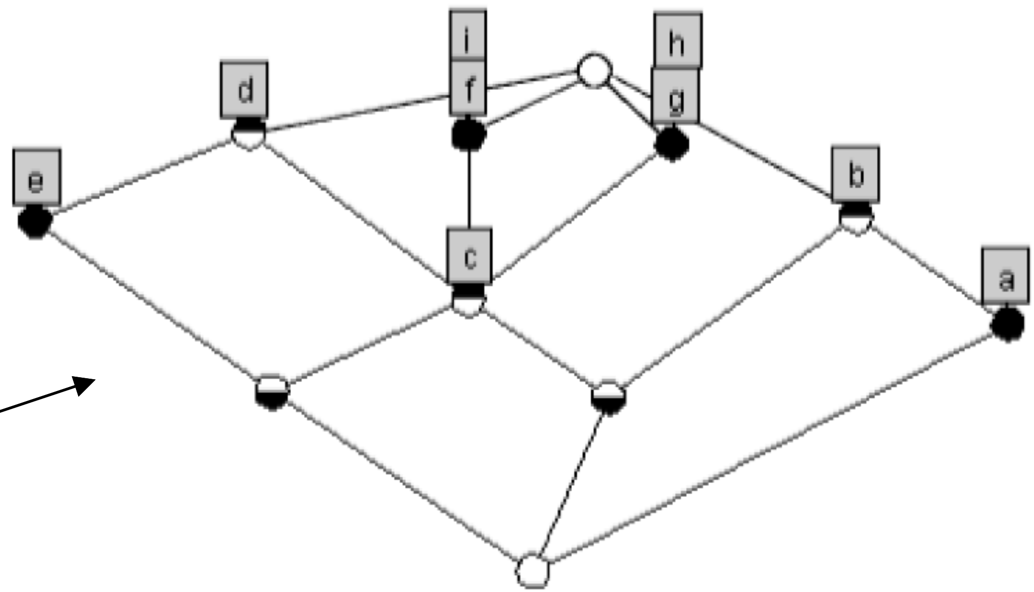
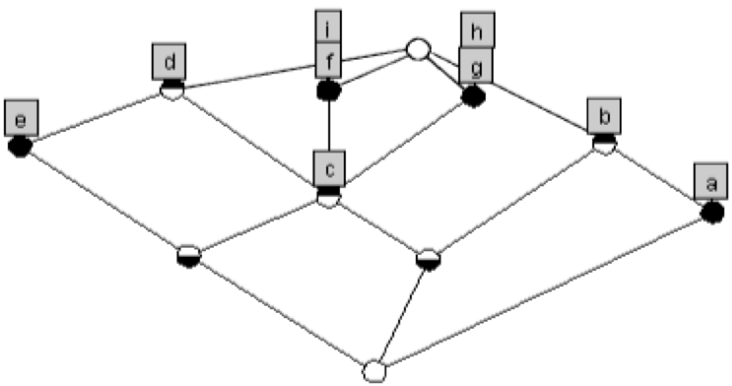


$\underline{\mathcal{B}}(\Phi,\mathcal{A},\ni)$

$$\mathcal{A}=\bigcup_{\phi\in\Phi}\phi$$

S-encodability and planar formal concept lattices

If Φ is S-encodable, then the formal concept lattice $\underline{\mathcal{B}}(\Phi, \mathcal{A}, \ni)$ is planar



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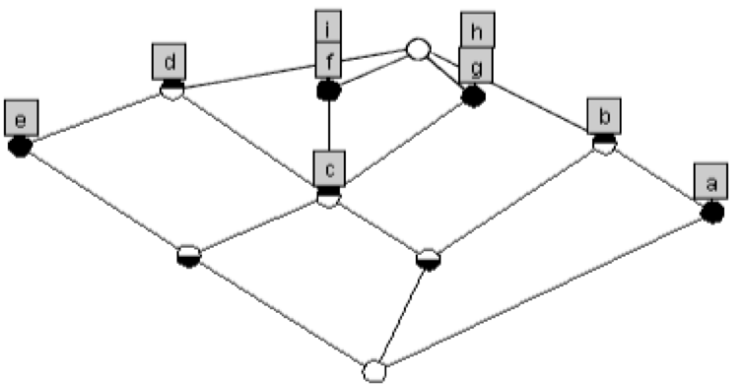
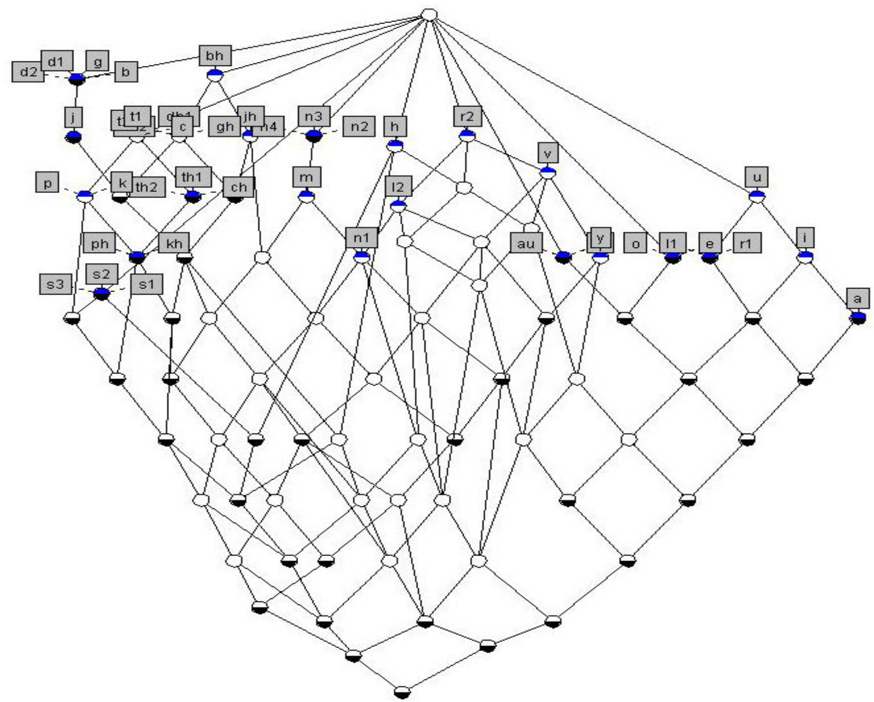
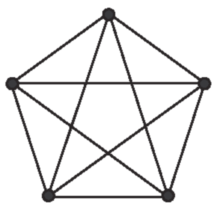




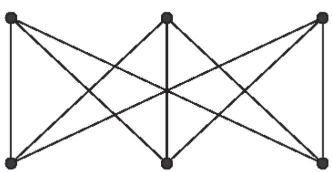
concept lattice for Pāṇini's phonological classes

S-encodability and planar formal concept lattices

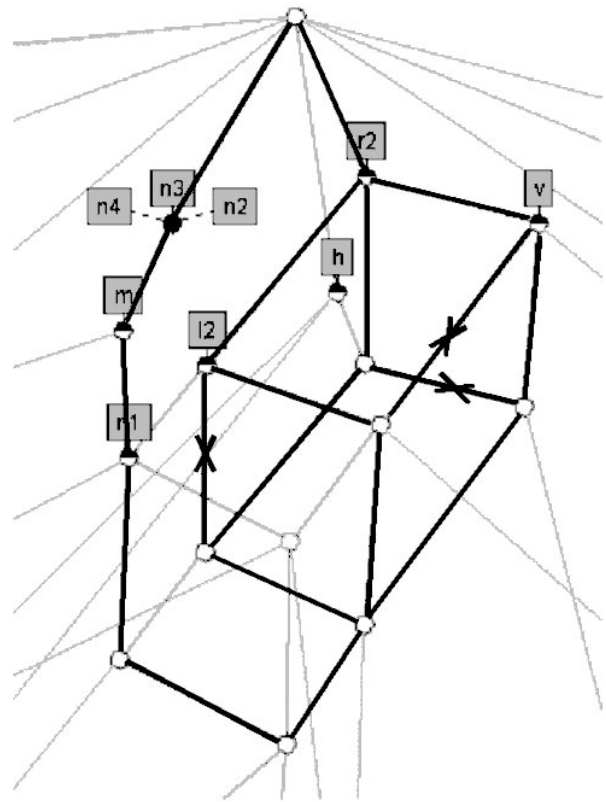
Criterion of Kuratowski:
A graph is planar iff it has neither K^5 nor $K_{3,3}$ as a *minor*.



K^5



$K_{3,3}$



part of the concept lattice for Pāṇini's phonological classes

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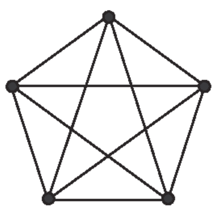
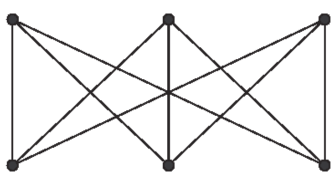


K^5



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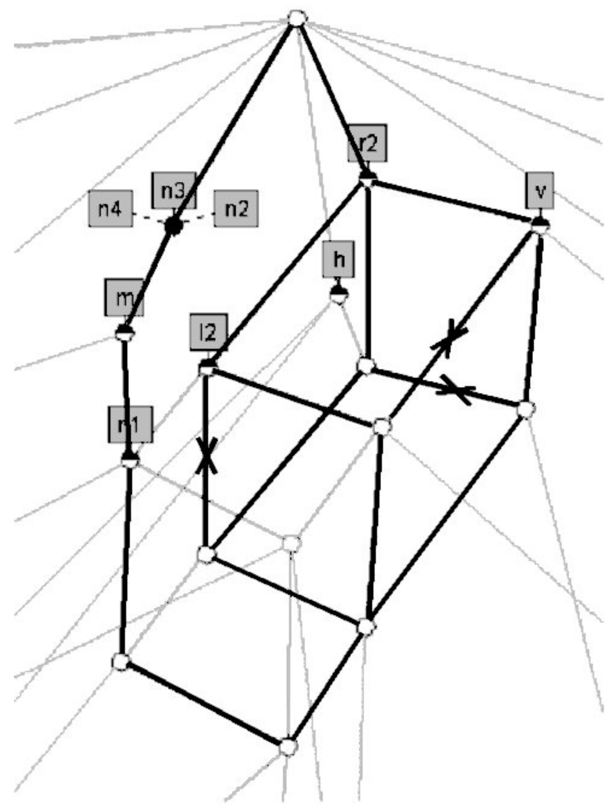
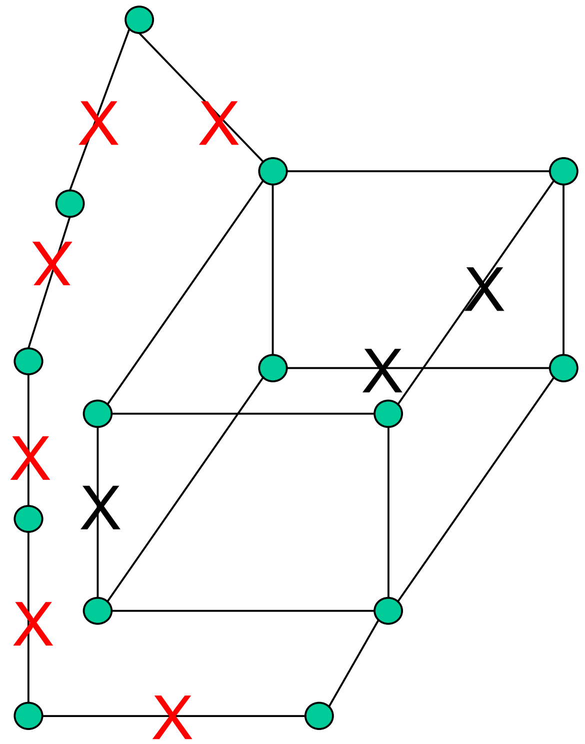


part of the concept lattice for Pāṇini's phonological classes

K^5 is a minor of the concept lattice for Pāṇini's phonological classes



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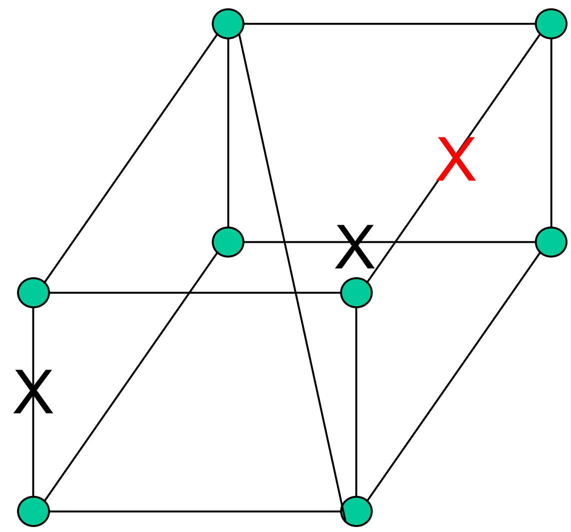

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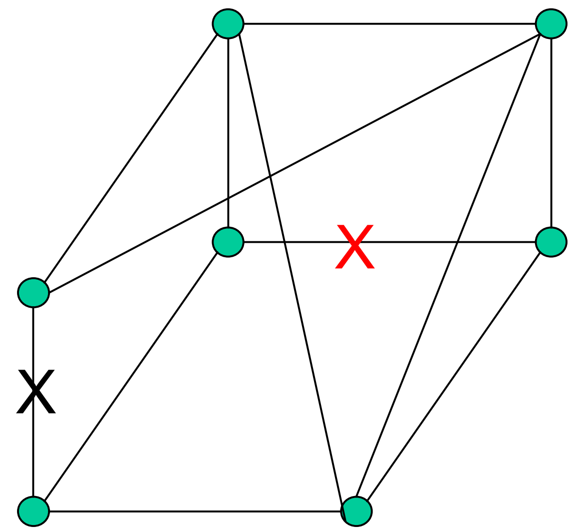

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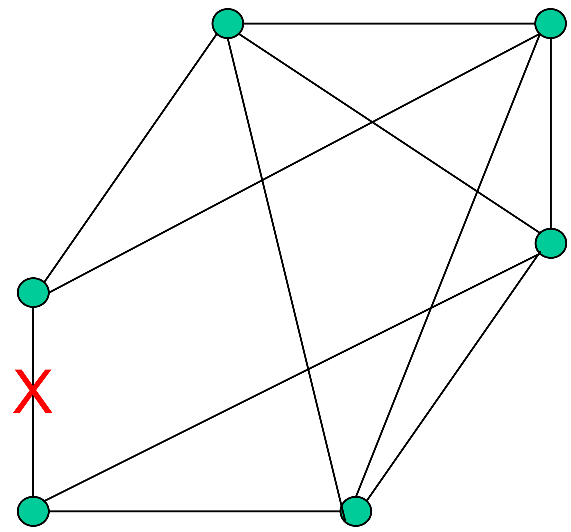

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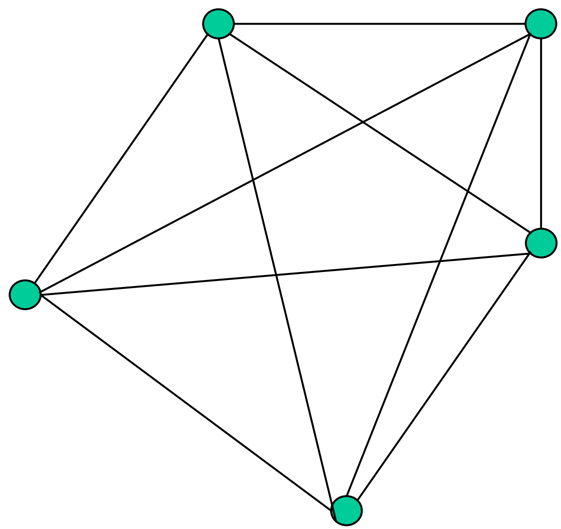
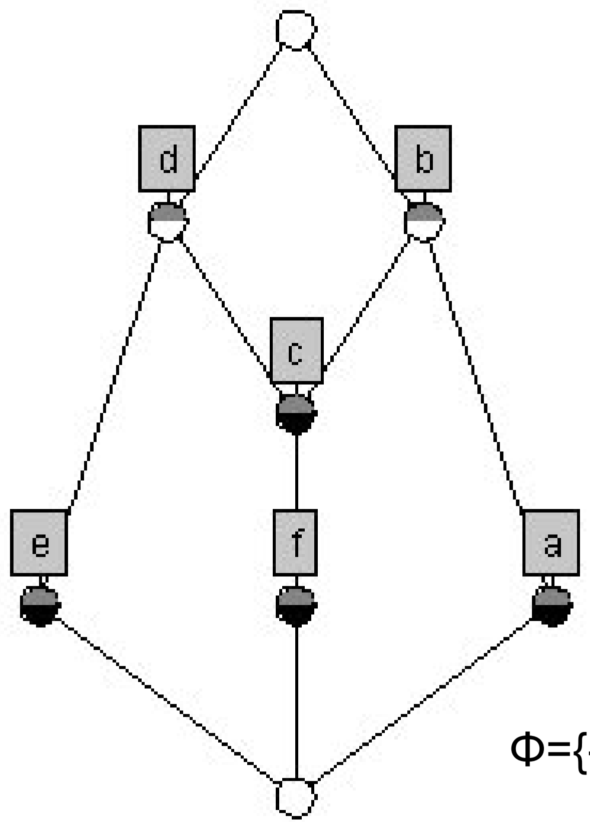


We are not done yet!



plane but not S-encodable!

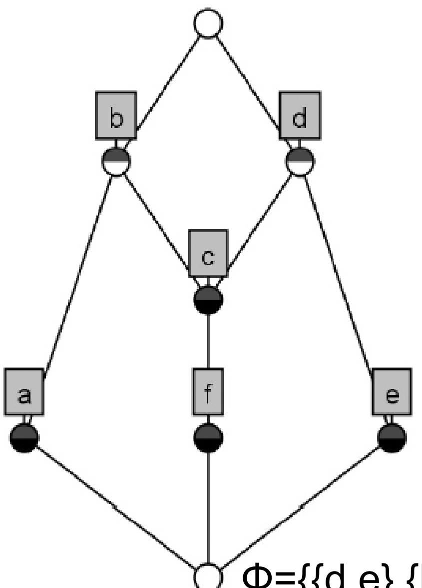
Φ={{d,e},{b,c,d,f,},{a,b},{b,c,d}}

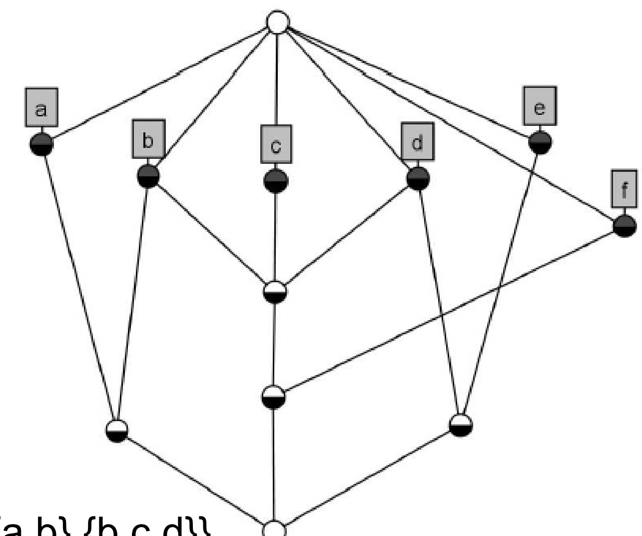
Existence of S-alphabets

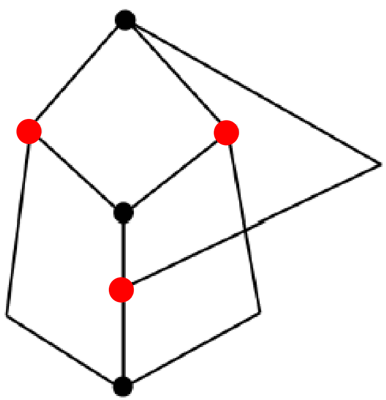
$\bar{\Phi} = \Phi \cup \{\{a\} : a \in \mathcal{A}\}$

The following statements are equivalent:

1. Φ is S-encodable
2. $\underline{\mathcal{B}}(\bar{\Phi}, \mathcal{A}, \ni)$ is planar







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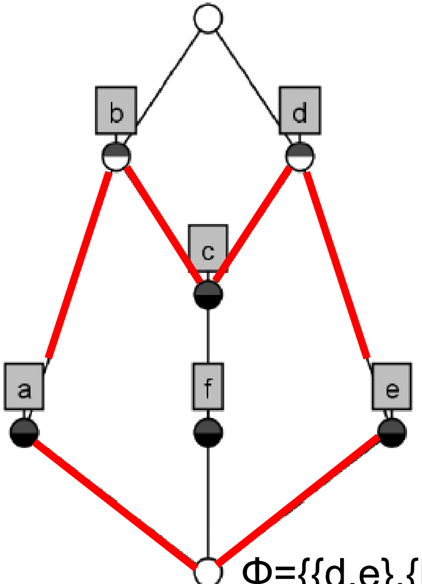

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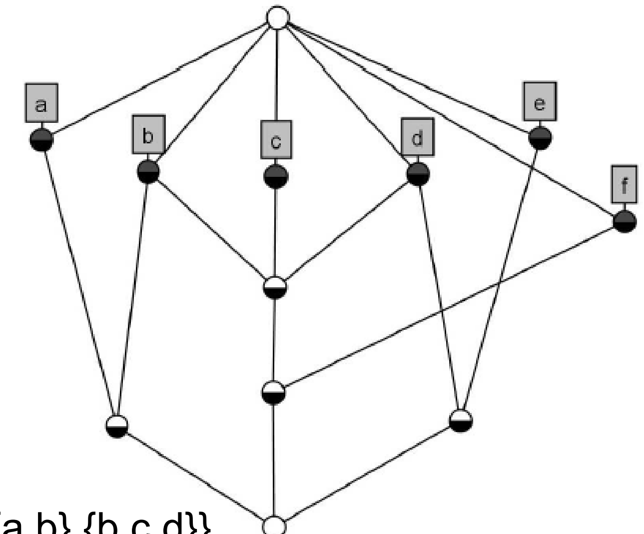

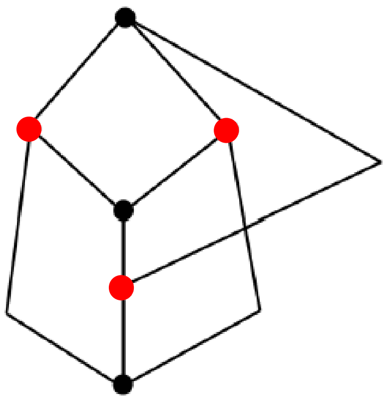

Existence of S-alphabets

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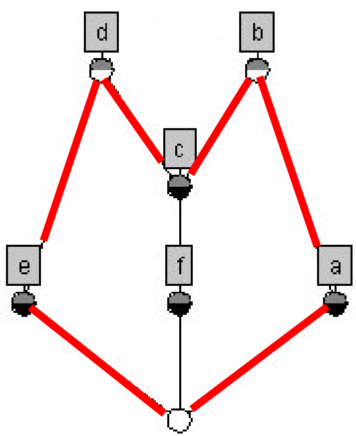
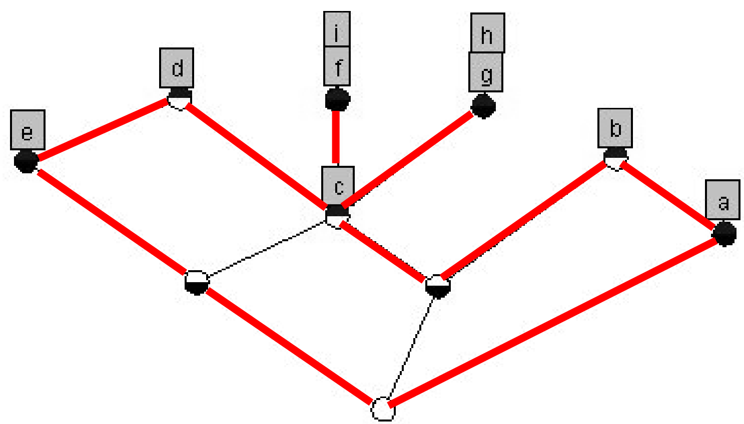
The following statements are equivalent:

1. Φ is S-encodable
2. $\underline{\mathcal{B}}(\bar{\Phi}, \mathcal{A}, \ni)$ is planar
3. the S-graph contains all attribute concepts

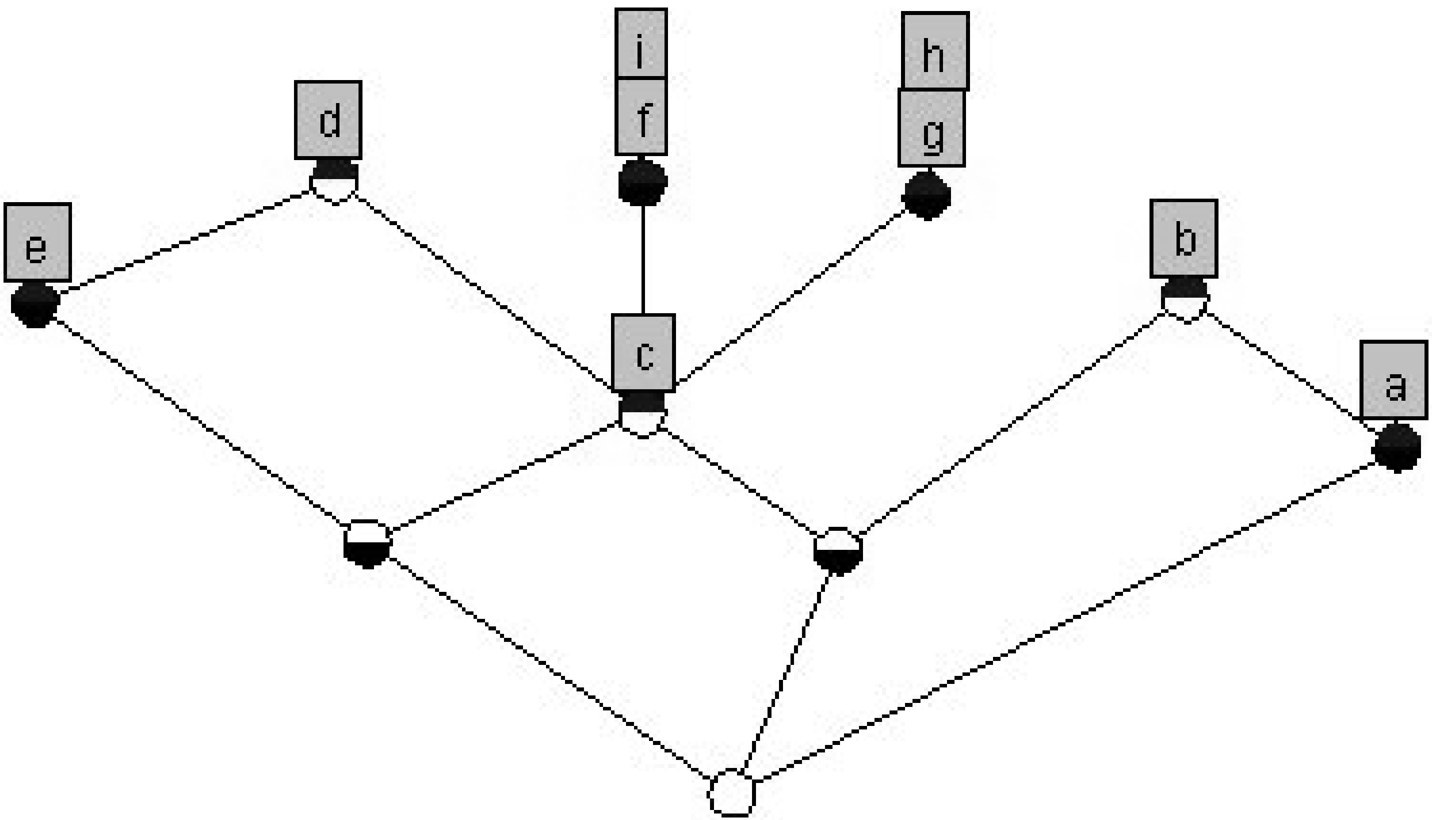


not S-encodable



S-encodable

Construction of S-alphabets



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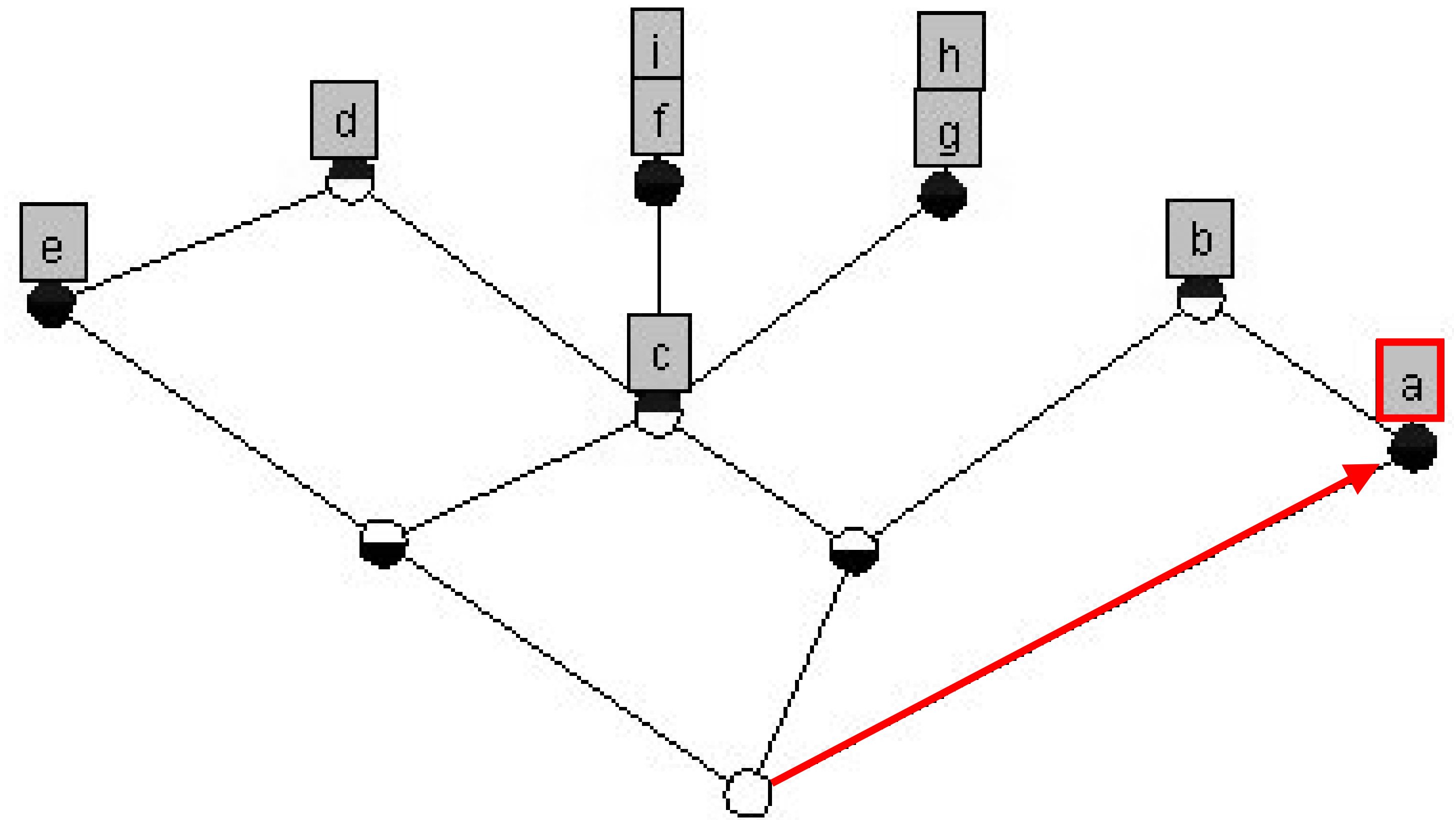
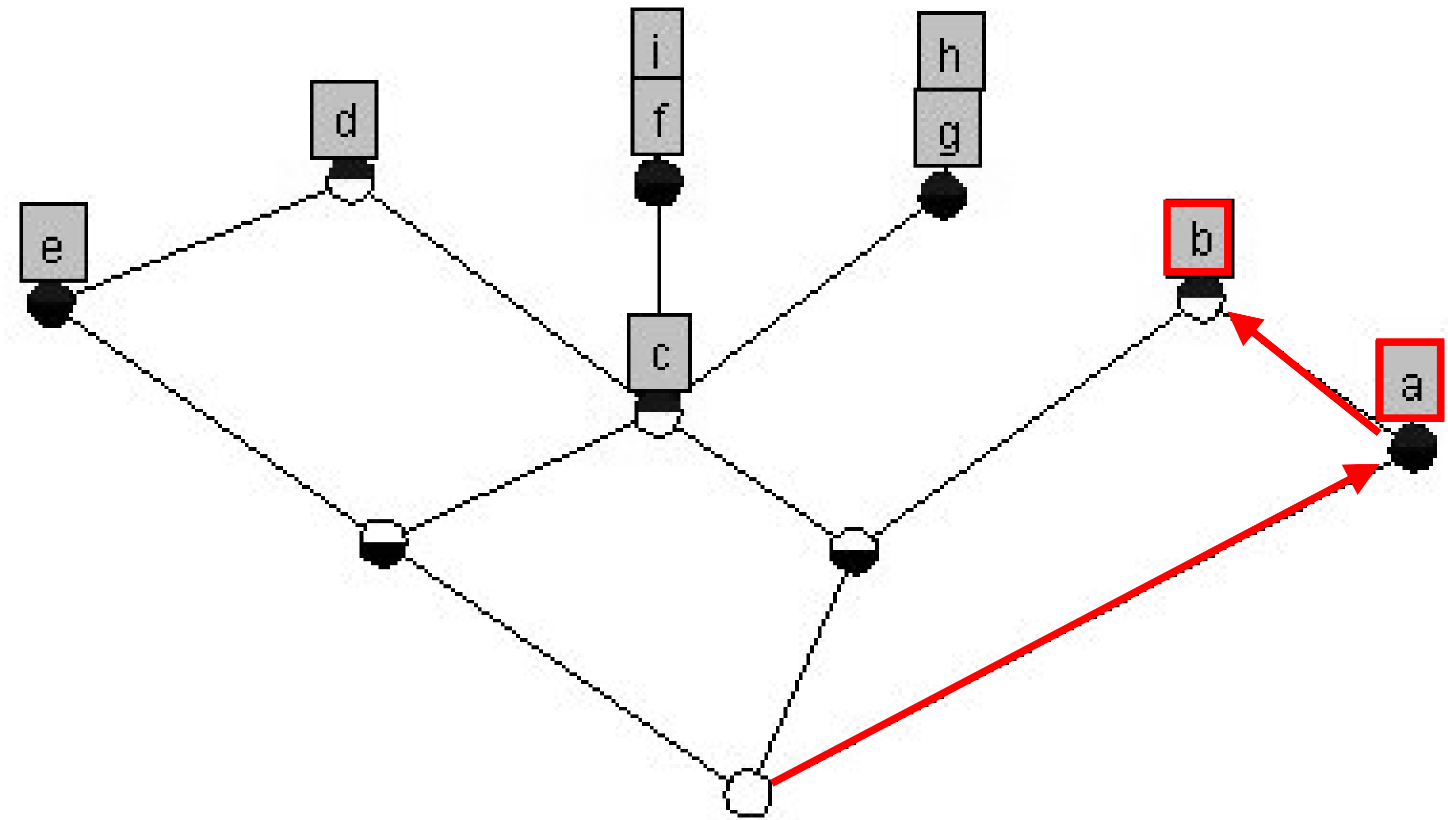


a

Construction of S-alphabets



a b

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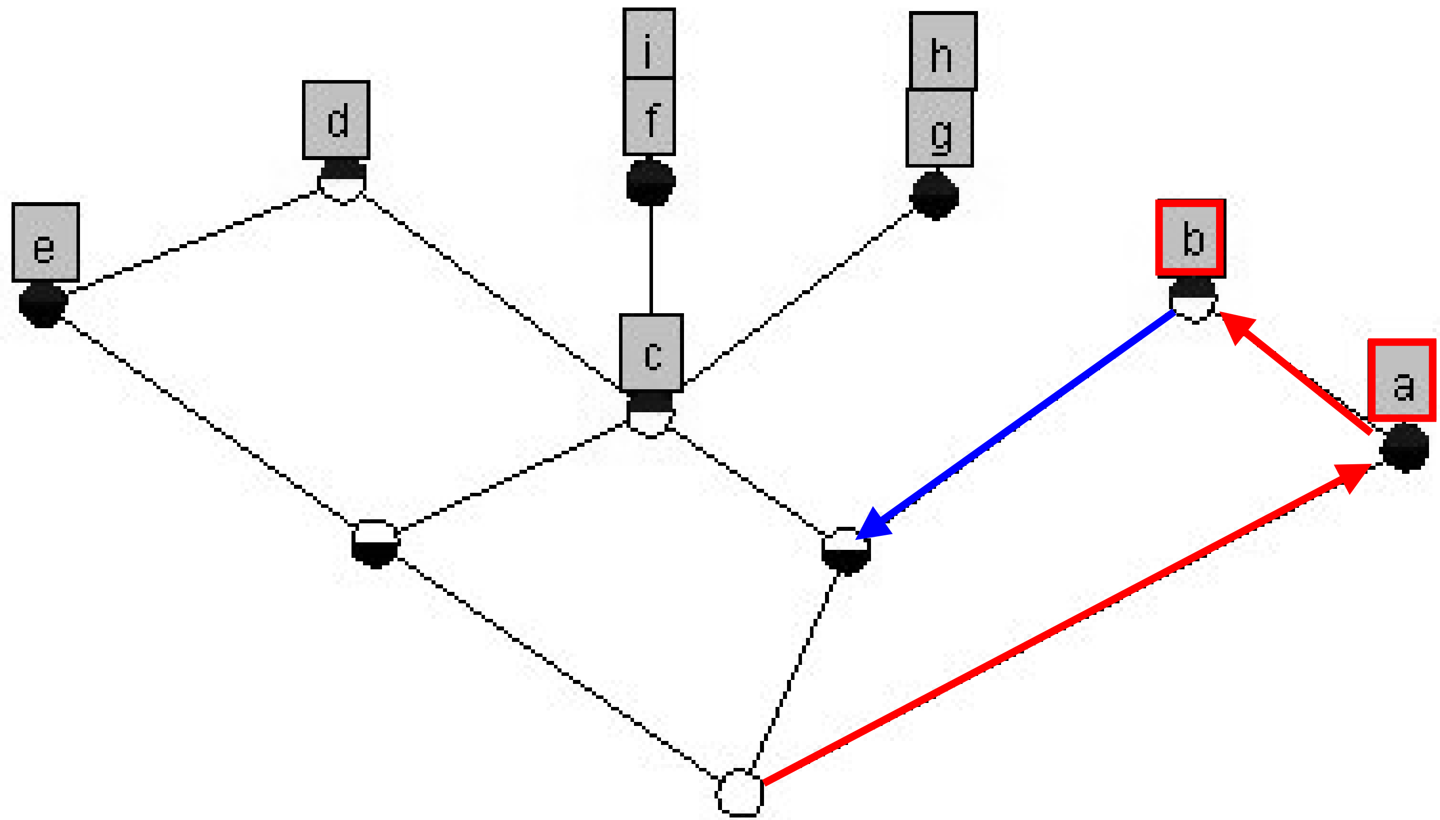
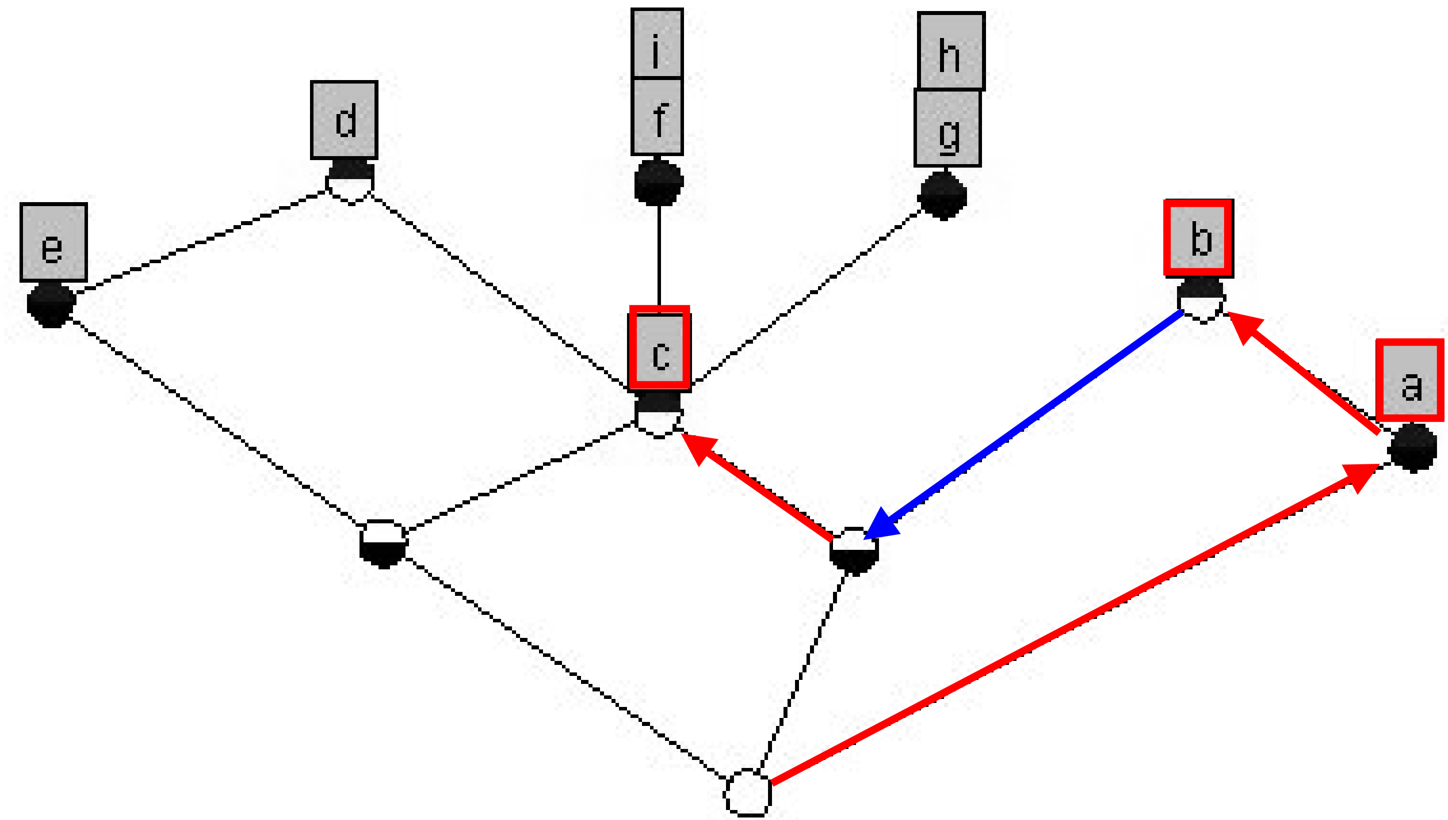


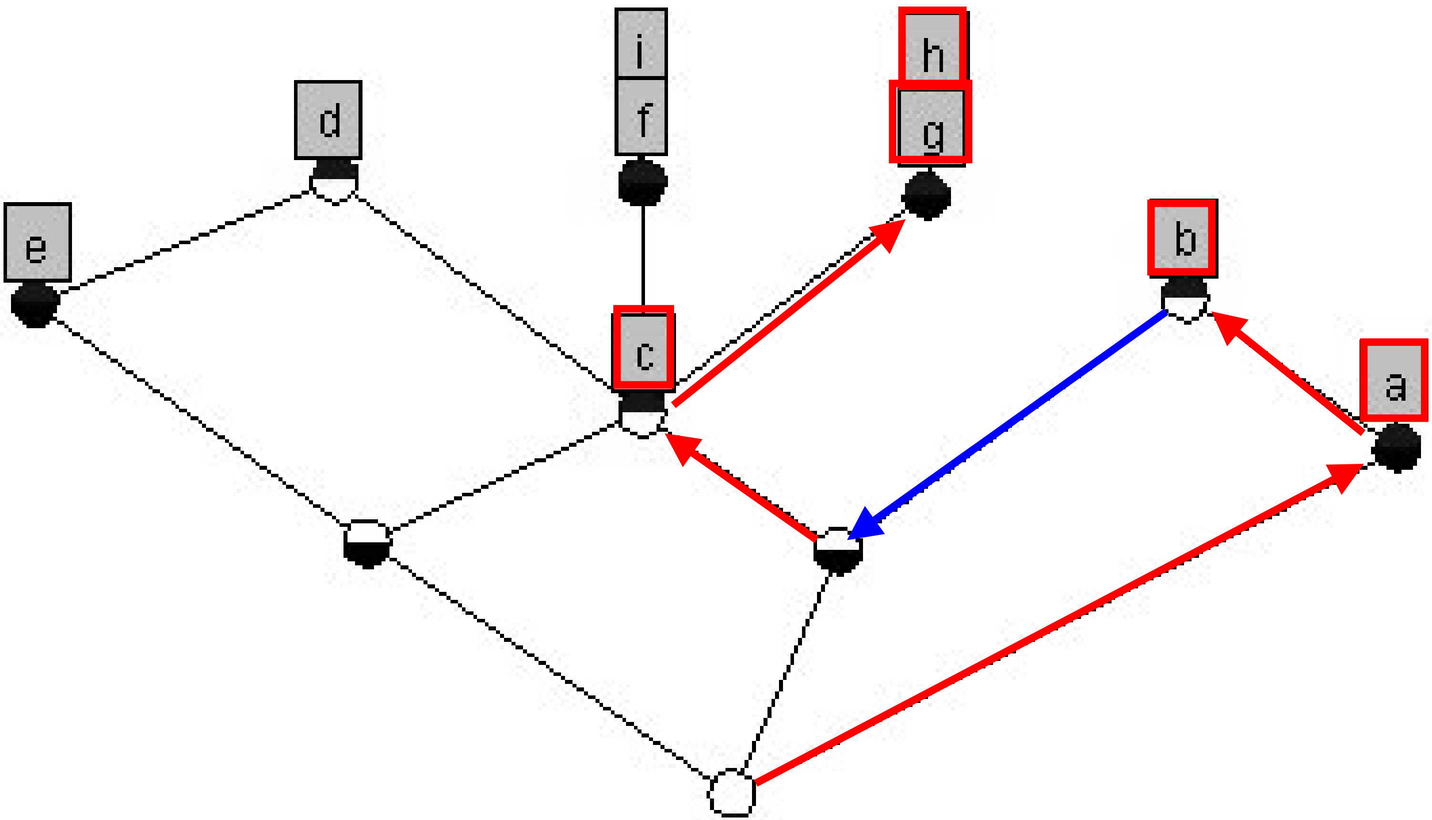
a b **M$_1$**

Construction of S-alphabets



a b **M$_1$** c

Construction of S-alphabets



a b **M$_1$** c g h

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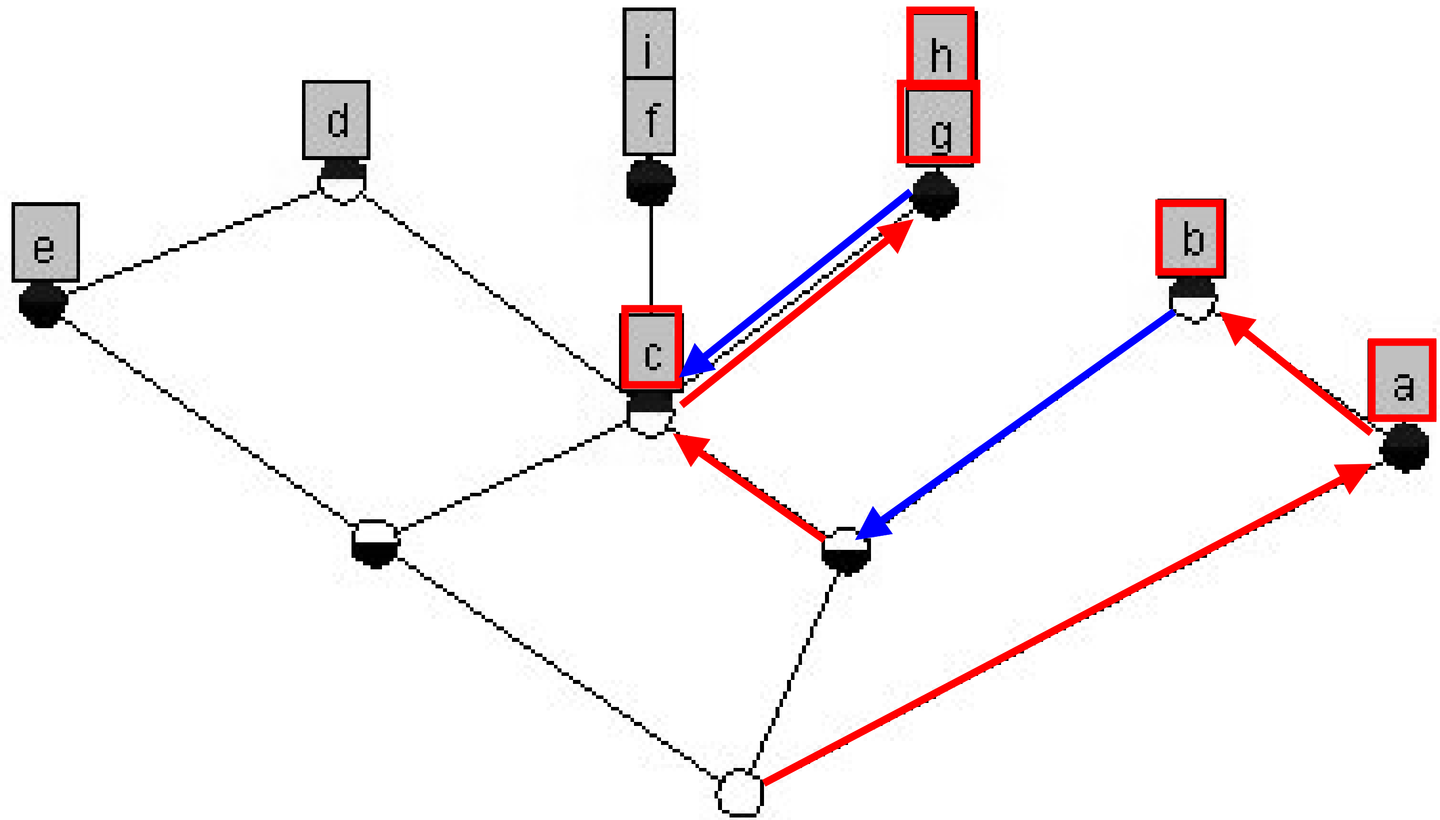

a b $\mathbf{M_1}$ c g h $\mathbf{M_2}$ c

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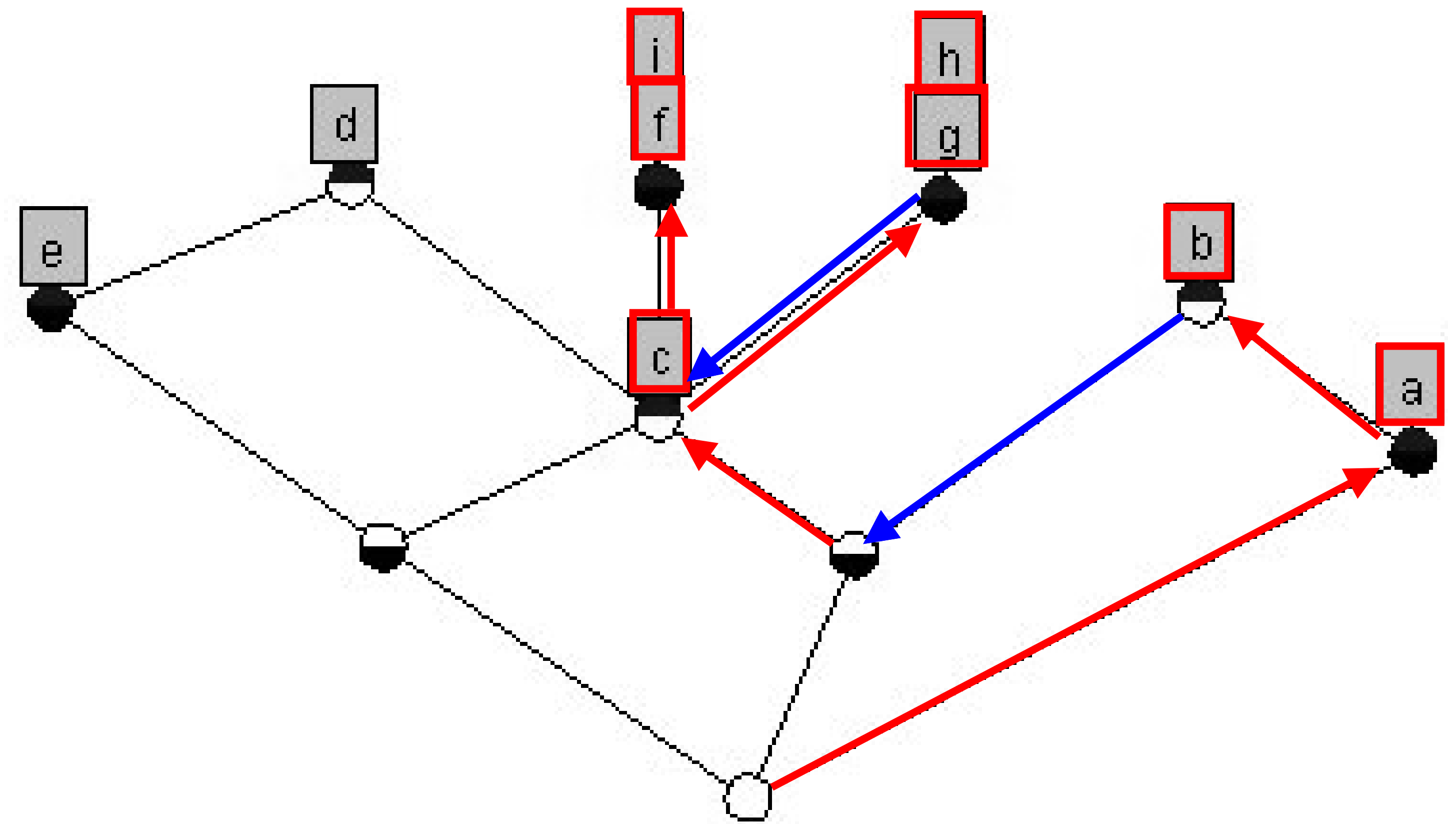

a b $\mathbf{M_1}$ c g h $\mathbf{M_2}$ c i f

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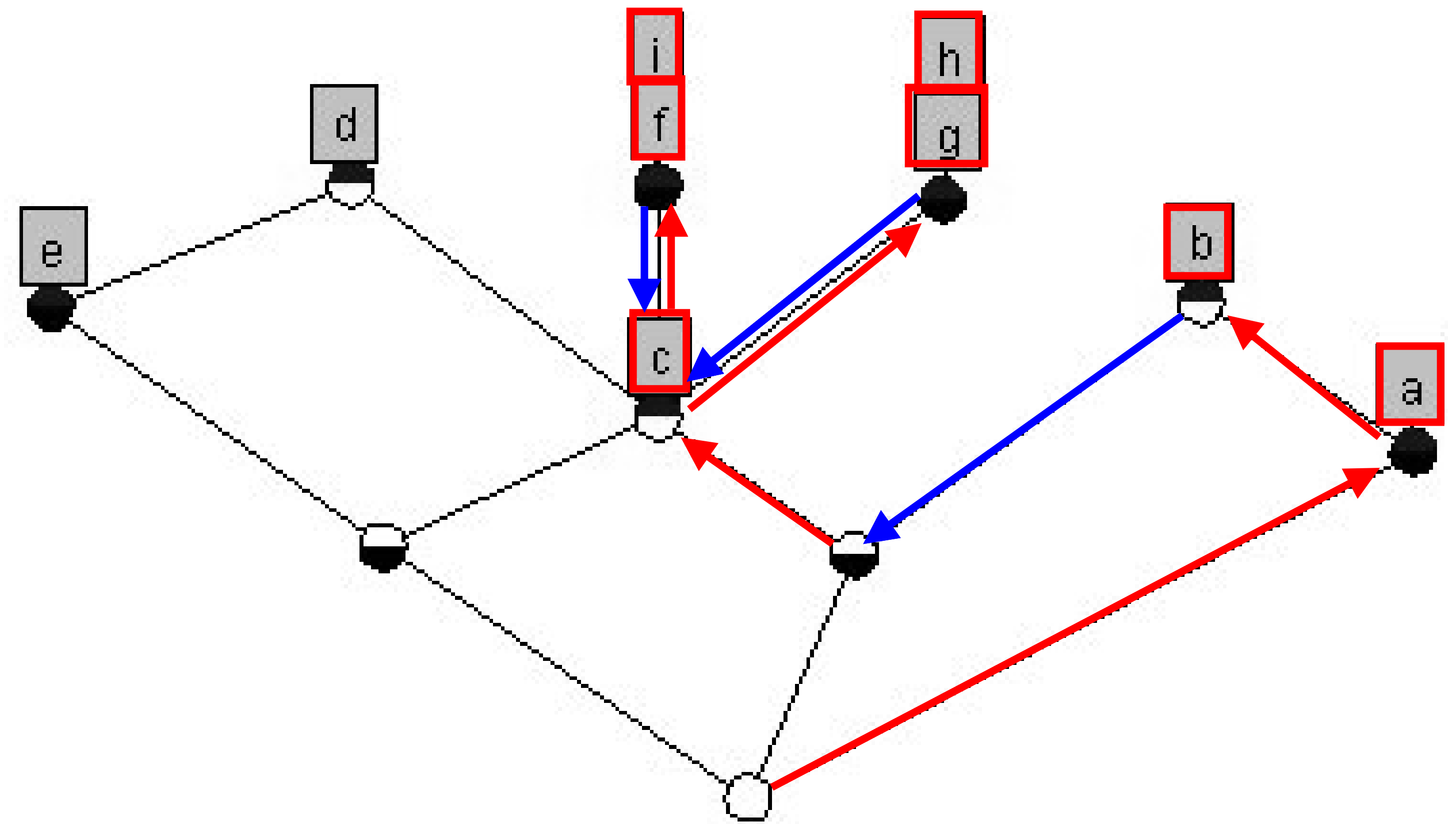

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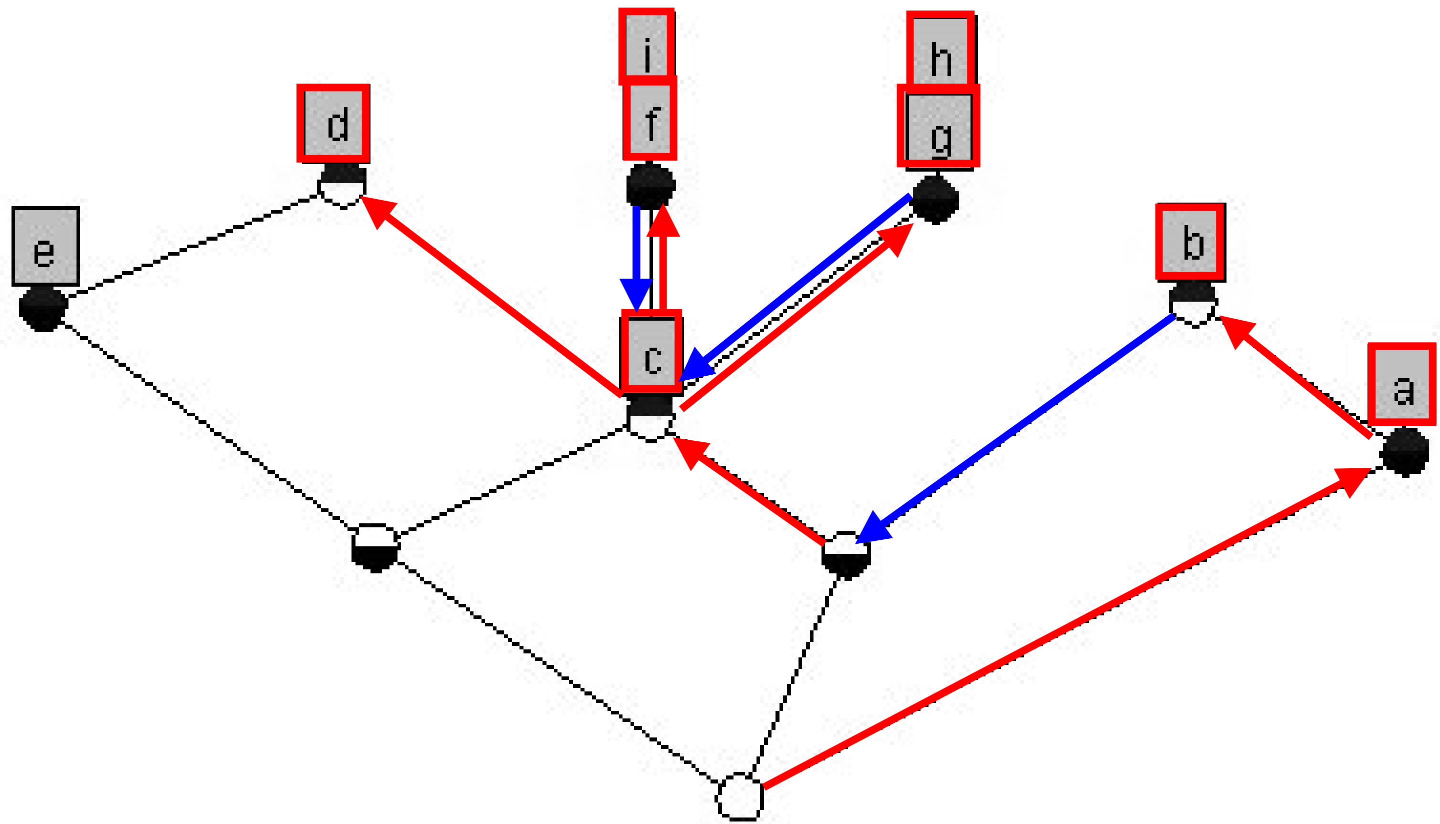
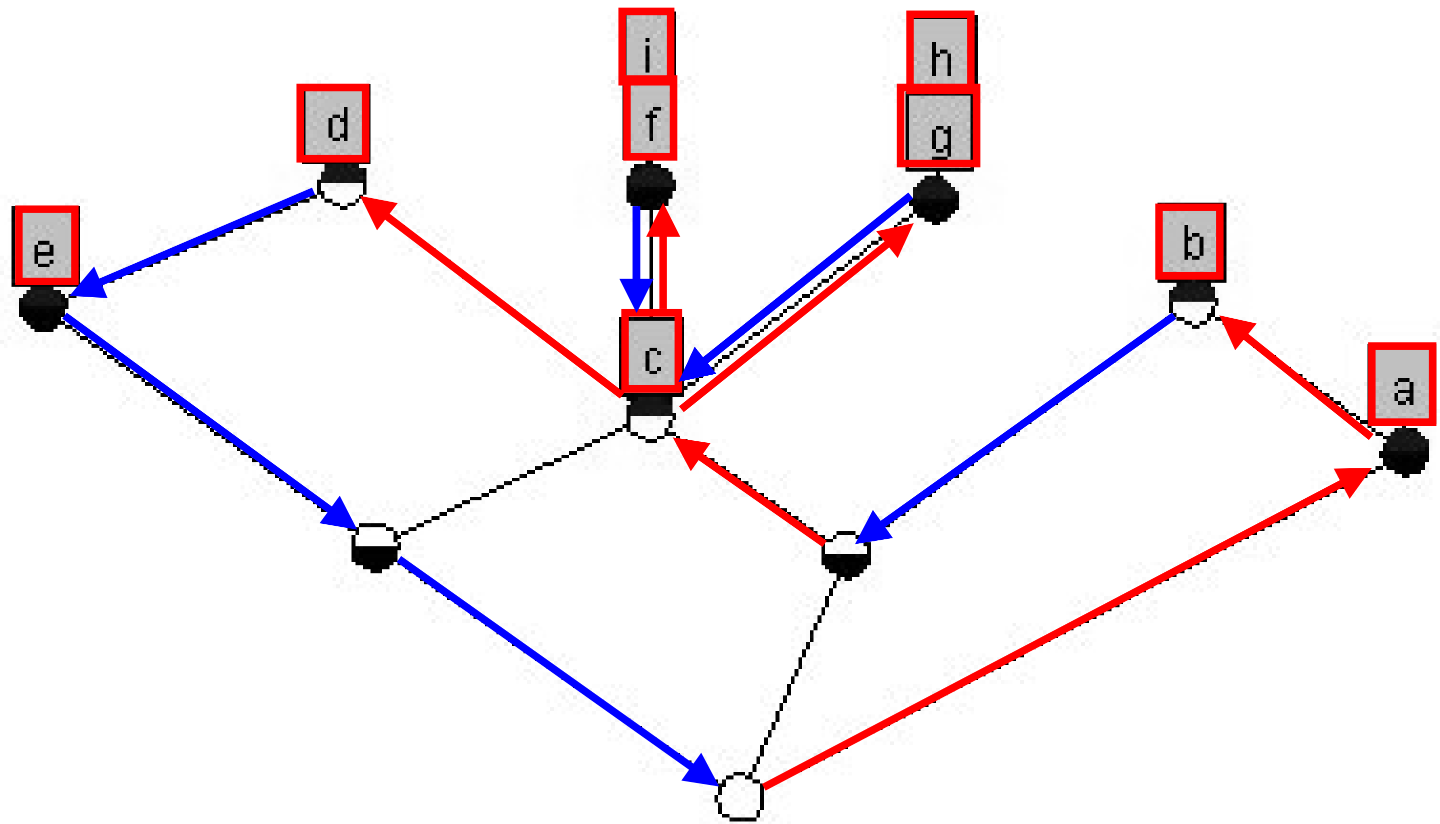


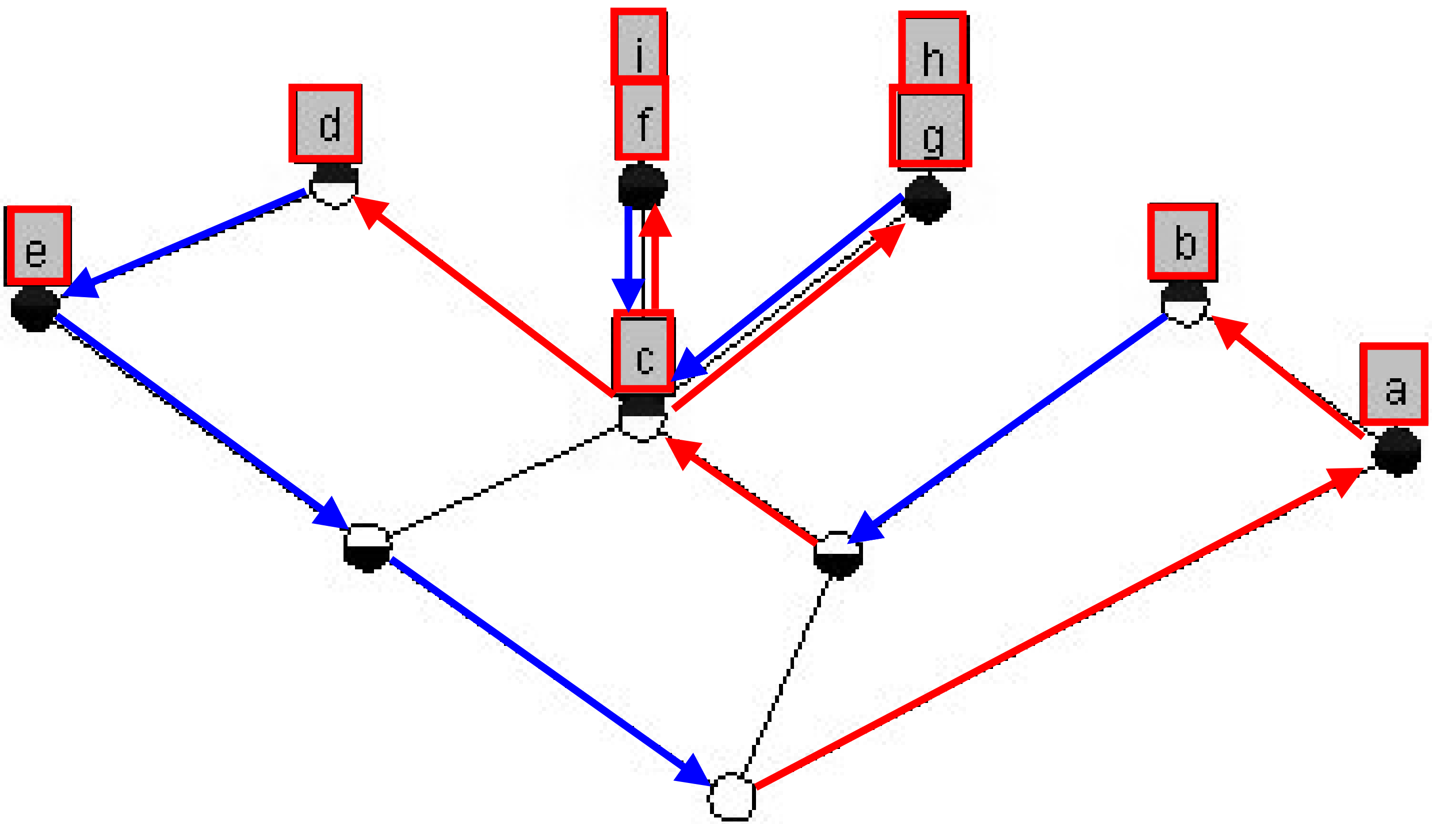
a b $\mathbf{M_1}$ c g h $\mathbf{M_2}$ c i f $\mathbf{M_3}$ c d

Construction of S-alphabets



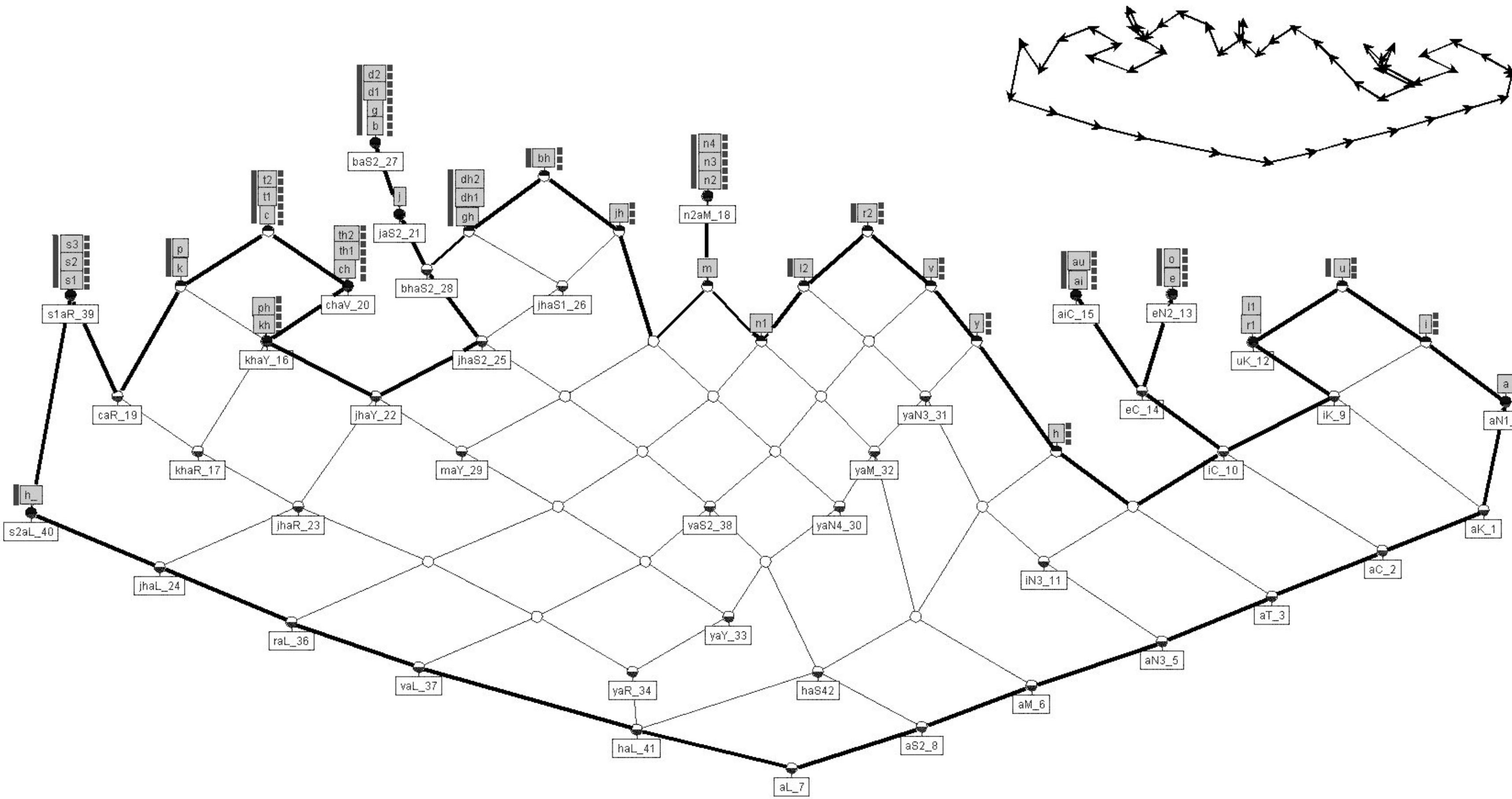
a b $\mathbf{M_1}$ c g h $\mathbf{M_2}$ c i f $\mathbf{M_3}$ c d $\mathbf{M_4}$ e $\mathbf{M_5}$

Construction of S-alphabets



a b $\mathbf{M_1}$ c g h $\mathbf{M_2}$ $\cancel{c}$ i f $\mathbf{M_3}$ $\cancel{c}$ d $\mathbf{M_4}$ e $\mathbf{M_5}$

Pāṇini's Śivasūtras are optimal



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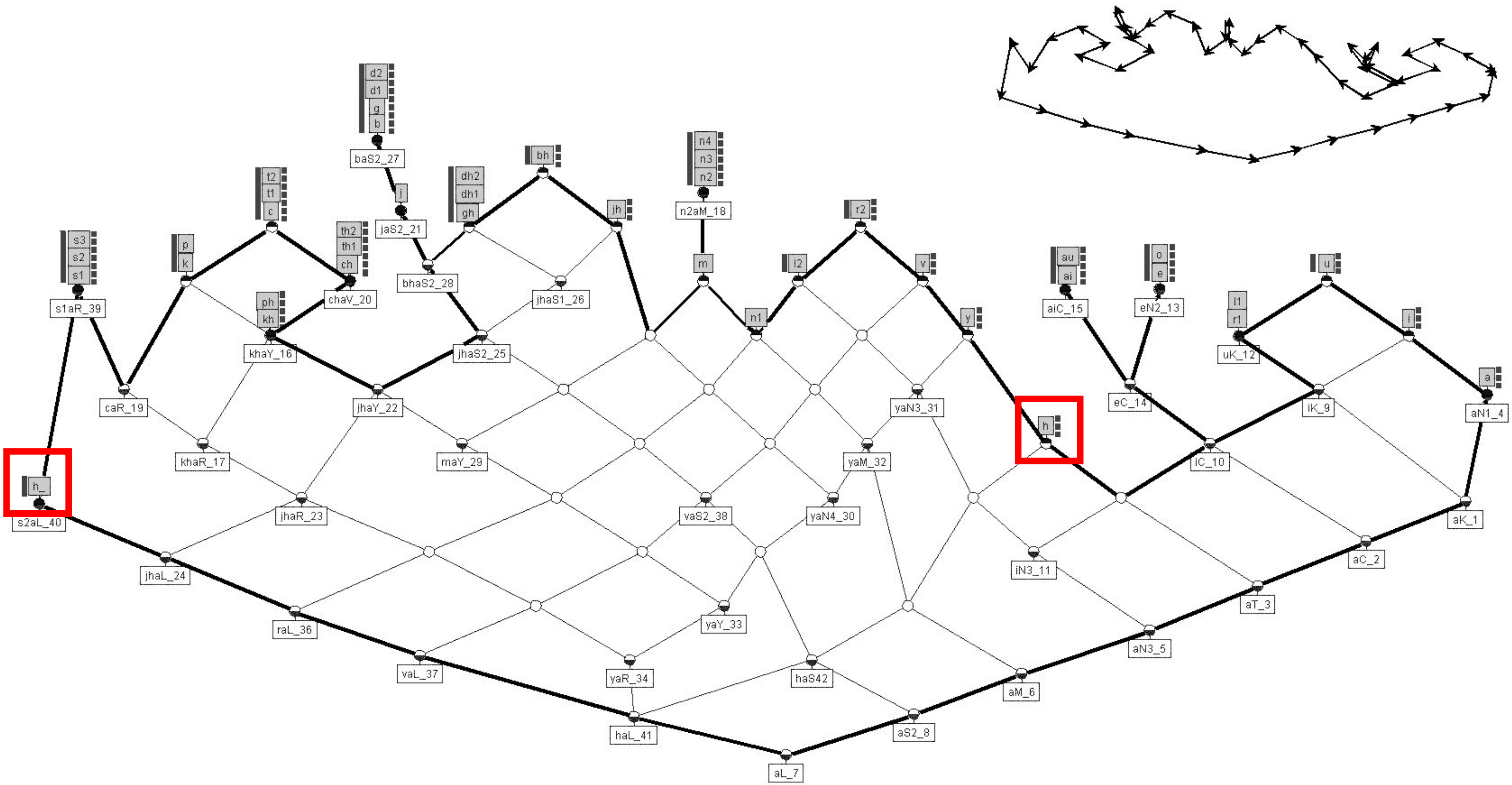

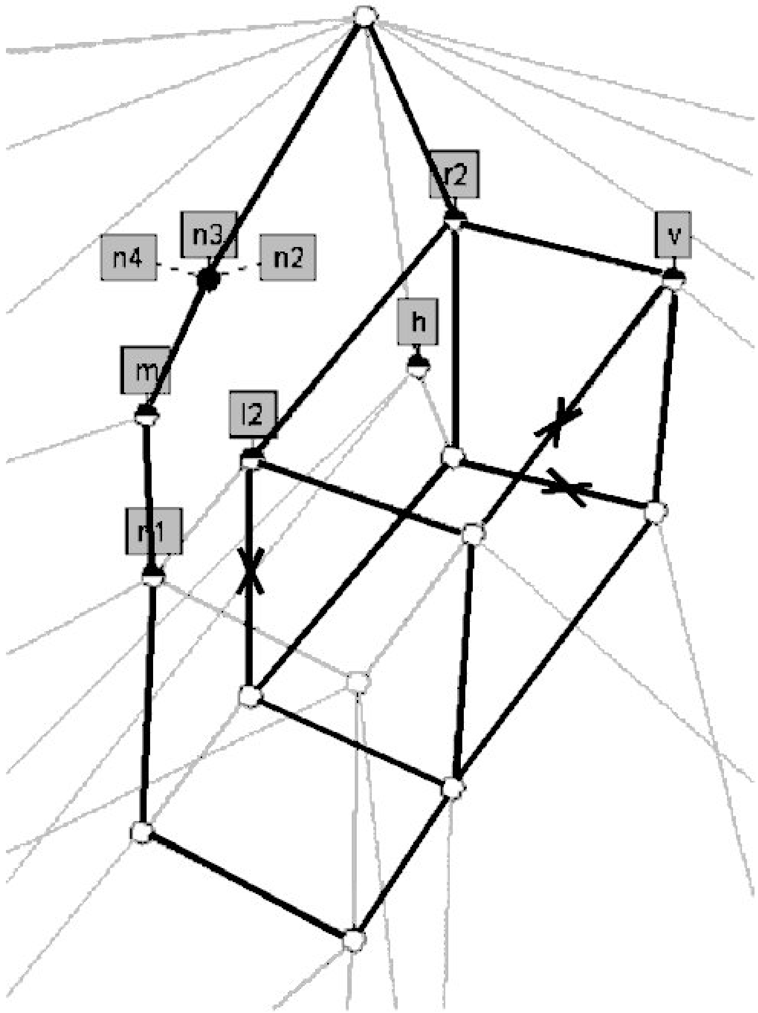

r2
v
n3
n4
n2
h
m
l2
n1

Pāṇini's Śivasūtras are optimal