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A Mathematical Analysis of Pāṇini's *Śivasūtras*

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Abstract. In Pāṇini's grammar of Sanskrit one finds the *Śivasūtras*, a table which defines the natural classes of phonological segments in Sanskrit by intervals. We present a formal argument which shows that, using his representation method, Pāṇini's way of ordering the phonological segments to represent the natural classes is optimal. The argument is based on a strictly set-theoretical point of view depending only on the set of natural classes and does not explicitly take into account the phonological features of the segments, which are, however, implicitly given in the way a language clusters its phonological inventory. The key idea is to link the graph of the Hasse-diagram of the set of natural classes closed under intersection to *Śivasūtra*-style representations of the classes. Moreover, the argument is so general that it allows one to decide for each set of sets whether it can be represented with Pāṇini's method. Actually, Pāṇini had to modify the set of natural classes to define it by the *Śivasūtras* (the segment *h* plays a special role). We show that this modification was necessary and, in fact, the best possible modification. We discuss how every set of classes can be modified in such a way that it can be defined in a *Śivasūtra*-style representation.[1]

Key words: Pāṇini, *Śivasūtras*, representation of natural classes

1. Pāṇini's *Śivasūtras*

Among grammars, Pāṇini's description of Sanskrit takes up an outstanding position. On the one hand, it is one of the oldest recorded grammars but also one of the most complete grammars of any language. The text of the grammar is rather accurately preserved, as it consists of *sūtras* designed for ritual repetition. On the other hand, this grammar attracts attention because it anticipates many structural and methodological approaches of modern linguistic theories.

> "Modern linguistics acknowledges it as the most complete generative grammar of any language yet written, and continues to adopt technical ideas from it." (Kiparsky, 1994)

Or, as Bloomfield states about this grammar:

> "The descriptive grammar of Sanskrit, which Pāṇini brought to its perfection, is one of the greatest monuments of human intelligence and an indispensable model for the description of languages." (Bloomfield, 1929)

Table I. Pāṇini's *Śivasūtras.*

1.	a	i	u			Ṇ
2.				ṛ	ḷ	K
3.		e	o			Ṅ
4.		ai	au			C
5.	h	y	v	r		Ṭ
6.					l	Ṇ
7.	ñ	m	ṅ	ṇ	n	M
8.	jh	bh				Ñ
9.			gh	ḍh	dh	Ṣ
10.	j	b	g	ḍ	d	Ś
11.	kh	ph	ch	ṭh	th	
			c	ṭ	t	V
12.	k	p				Y
13.		ś	ṣ	s		R
14.	h					L

Table II. Example of a *pratyāhāra:* iC = {i, u, ṛ, ḷ, e, o, ai, au}.

1.	a	**i**	**u**			Ṇ
2.				**ṛ**	**ḷ**	K
3.		**e**	**o**			Ṅ
4.		**ai**	**au**			C
5.	h	y	v	r		Ṭ

The grammar gives a theoretical analysis of classical Sanskrit as spoken by the priestly class at the time of its formulation (according to Kiparsky (1994), around 350 BC). As a whole, the grammar consists of four parts, of which the *Aṣṭādhyāyī* plays the central role as it contains nearly 4000 rules governing how the elements determined in the other components can be used. The *Śivasūtras* form the component in which the phonological segments of the language and their grouping in natural phonological classes, designated by *pratyāhāras*, is defined (a short survey of the structure of Pāṇini's grammar can be found in Kiparsky (1994)). In the *Aṣṭādhyāyī* Pāṇini refers to the phonological classes in hundreds of rules.

The *Śivasūtras* identify 42 phonological segments and consist of 14 *sūtras* (rows in Table I), each of which consists of a sequence of phonological segments (transcribed with small letters) bounded by a marker (transcribed with a capital letter), called *anubandha.* Phonological classes are denoted by abbreviations, called *pratyāhāras*, consisting of a phonological segment and an *anubandha.* The elements of such a class are defined by the *Śivasūtras* given in Table I and are the continuous sequence of phonological segments starting with the given segment and ending with the last segment before the *anubandha.* Table II gives an example of a *pratyāhāra.*

A typical phonological rule found in the *Aṣṭādhyāyī* is **iko yaṇ aci**, the functional analysis of which is $[\text{ik}]_{\text{GEN}}[\text{yaṇ}]_{\text{NOM}}[\text{ac}]_{\text{LOC}}$. The case markers are used meta-linguistically and denote the role that an expression marked by a case suffix plays in the rule. The technical expressions **ik**, **yaṇ** and **ac** belong to the meta-language, too; they stand for the *pratyāhāras* iK, yṆ and aC. The vowel 'a' in the expression **yaṇ** fulfills two tasks: first, it serves as a linking vowel which turns the *pratyāhāra* into a pronounceable syllable, and second, it prevents the consonant 'y' from being mistaken for the anubandha 'Y'. The rule **iko yaṇ aci** is interpreted as iK → [yṆ]/_[aC]. This rule states that the vowels of the class iK = {i, u, r̥, l̥} are replaced by their nonsyllabic counterparts yṆ = {y, v, r, l} before a vowel aC = {a, i, u, r̥, l̥, e, o, ai, au}.

Altogether, 281 *pratyāhāras* can be constructed,[2] which is more than the 42 actually referred to by rules of the *Aṣṭādhyāyī*, but it is still a small number compared with the number of all classes that can be formed from the phonological segments, which is $2^{42} > 4 \cdot 10^{12}$.

As Kornai (1993) points out clearly, the task of characterizing a phonological system of a language is to specify the segmental inventory, phonological rules, and the set of natural classes of phonological segments to enable generalized rules.

> "What is required is a clever notation that lets us characterize any such $R \subset S$, traditionally called *a natural class*, in a compact manner so that rules in terms of natural classes are just as easy, or perhaps even easier, to deal with as rules stated in terms of segments." (Kornai, 1993)

The set of natural classes is externally given by the phonological patterning of a language and it always meets two conditions: first, it is small compared to the set of unnatural classes and second, the set of natural classes is basically closed under intersection. Kornai (1993) stipulates that these conditions have to be reflected in a notational device for the classes, too.

Kornai stresses that the representation device used for the notation of the classes must make it easier to use natural classes than unnatural ones (e.g. the complement of a natural class is generally unnatural). Contemporary phonological theories build up a structured system of phonological features that are used to characterize the natural classes. Instead of referring to phonological features to define a phonological class, Pāṇini refers to intervals in a linear order of the phonological segments. His method of defining the natural classes by *pratyāhāras*, denoting intervals of the *Śivasūtras*, meets the required conditions (as does the contemporary method).

The phonological classes of a grammar are mutually related: classes can be subclasses of other classes, two or more classes can have common elements, etc. These connections are naturally represented in a hierarchy. A *Śivasūtra*-style representation encodes such connections in a linear form.[3] The linear representation method is so ingenious that it allows one to answer questions about the hierarchical relations of the classes without referring to all the elements of the classes. For example, two classes have common elements if at least one interval boundary of the first lies in the interval of the second.[4] An aim of this paper is to determine the conditions under which a set of sets in fact has a *Śivasūtra*-style linear representation.

The rest of the paper is organized as follows: In Section 2 a general formalization of Pāṇini's *Śivasūtra*-style representation of phonological classes is given. Furthermore, the main questions which will be answered in the course of the paper are raised. Section 3 explains how the Hasse-diagrams of sets of subsets determine whether a *Śivasūtra*-style representation of natural classes exists. Since some results of graph theory are needed, a brief introduction to planar graphs is given. Finally, in Section 4 a procedure is presented that constructs a good *Śivasūtra*-style representation of a set of natural classes if it exists. This section ends with the proof that Pāṇini has chosen an optimal *Śivasūtra*-style representation. Furthermore, examples of feature-based analyses are stated and translated into *Śivasūtra*-style representations. The whole approach is based only on a set- and order-theoretical investigation of the set of natural classes used in Pāṇini's grammar of Sanskrit. No external – phonological or methodological – arguments are involved.

2. General Definitions and the Main Questions

DEFINITION 2.1. A well-formed *Śivasūtra*-alphabet *(short* S-alphabet*) is a triple* $(\mathcal{A}, \Sigma, <)$ *consisting of a finite object alphabet* $\mathcal{A}$*, a finite set of markers* Σ *(such that* $\mathcal{A} \cap \Sigma = \emptyset$*), and a total order* $<$ *on* $\mathcal{A} \cup \Sigma$*.*

DEFINITION 2.2. *A subset* T *of the alphabet* $\mathcal{A}$ *is* S-encodable in an S-alphabet $(\mathcal{A}, \Sigma, <)$ *iff there exists* $a \in \mathcal{A}$ *and* $M \in \Sigma$*, such that* $T = \{b \in \mathcal{A} : a \leq b < M\}$*.* aM *is called the* pratyāhāra *or* S-encoding *of* T *in* $(\mathcal{A}, \Sigma, <)$.

The set of S-encodable sets in an S-alphabet $(\mathcal{A}, \Sigma, <)$ meets the two conditions of natural classes stipulated by Kornai: First, if $T_1, T_2 \subseteq \mathcal{A}$ are S-encodable in $(\mathcal{A}, \Sigma, <)$, with *pratyāhāras* a_1M_1 and a_2M_2, then $T_1 \cap T_2$ is S-encodable in $(\mathcal{A}, \Sigma, <)$, too, and the *pratyāhāra* of $T_1 \cap T_2$ is $\max(a_1, a_2)\min(M_1, M_2)$. Second, the set of S-encodable classes is small compared to the set of unnatural classes, since for an alphabet $\mathcal{A}$ consisting of n elements the number of *pratyāhāras* is at most $\binom{n}{2}$, while the number of all possible classes is $2^n - n$. Hence, the *pratyāhāras* form a suitable device for defining natural classes.

DEFINITION 2.3. *An S-alphabet* $(\mathcal{A}', \Sigma, <)$ corresponds *to a system of sets* $(\mathcal{A}, \Phi)$ *(where* Φ *is a set of subsets of* $\mathcal{A}$*) iff* $\mathcal{A} = \mathcal{A}'$ *and each element of* Φ *is S-encodable in* $(\mathcal{A}', \Sigma, <)$*. An S-alphabet which corresponds to* $(\mathcal{A}, \Phi)$ *is called an* S-alphabet *of* $(\mathcal{A}, \Phi)$*. A system of sets for which a corresponding S-alphabet exists is said to be* S-encodable.

For example, take the set of subsets

$$\Phi = \{\{d, e\}, \{b, c, d, f, g, h, i\}, \{a, b\}, \{f, i\}, \{c, d, e, f, g, h, i\}, \{g, h\}\} \quad (1)$$

of the alphabet $\mathcal{A} = \{a, b, c, d, e, f, g, h, i\}$: it is S-encodable and

$$a\, b M_1 c\, g\, h M_2\, f i M_3\; d\; M_4\; e\; M_5 \quad (2)$$

is one of the corresponding S-alphabets. The *pratyāhāras* of Φ are: dM_5, bM_4, aM_1, fM_3, cM_5 and gM_2.

DEFINITION 2.4. *An S-alphabet* $(\mathcal{A}, \Sigma, <)$ *of* $(\mathcal{A}, \Phi)$ *is said to be* good *iff there exists no other S-alphabet* $(\mathcal{A}, \Sigma', <')$ *of* $(\mathcal{A}, \Phi)$ *such that the set of markers* Σ' *has fewer elements than* Σ.

Looking at Pāṇini's *Śivasūtras* it is striking that the phonological segment *h* occurs twice, namely in *sūtra* 5 and *sūtra* 14. To model this phenomenon we will introduce the concept of *enlarging* an alphabet by duplicating some of its elements.[5]

$\hat{\mathcal{A}}$ is said to be an *enlarged alphabet* of $\mathcal{A}$ if there exists a surjective map $\vartheta : \hat{\mathcal{A}} \to \mathcal{A}$. We can extend the map ϑ naturally to sets ϑ: $\mathbf{P}(\hat{\mathcal{A}}) \to \mathbf{P}(\mathcal{A})$ with $\vartheta(\hat{\varphi}) = \{\vartheta(a) : a \in \hat{\varphi}\}$. It is clear that for every system of sets $(\mathcal{A}, \Phi)$ we can find an enlarged alphabet $\hat{\mathcal{A}}$ and a set of subsets $\hat{\Phi}$ of $\hat{\mathcal{A}}$ with $\Phi = \{\vartheta(\hat{\varphi}) : \hat{\varphi} \in \hat{\Phi}\}$ such that $(\hat{\mathcal{A}}, \hat{\Phi})$ is S-encodable.[6] To achieve such an S-encodable system of sets $(\hat{\mathcal{A}}, \hat{\Phi})$ we enlarge $\mathcal{A}$ and choose $\hat{\Phi}$ so that the elements of $\hat{\Phi}$ are disjoint. Then we arrange the sets of $\hat{\Phi}$ in a sequence and separate them by markers. The induced S-alphabet $(\hat{\mathcal{A}}, \hat{\Sigma}\, \hat{<})$ obviously corresponds to $(\hat{\mathcal{A}}, \hat{\Phi})$. The following example illustrates this procedure: Let

$$\Phi = \{\{a, b\}, \{a, c\}, \{b, c\}\} \text{ and } \mathcal{A} = \{a, b, c\} \quad (3)$$

be the system of sets for which an enlarged S-alphabet is searched. $(\mathcal{A}, \Phi)$ is not S-encodable because the elements of $\mathcal{A}$ cannot be linearly ordered in such a way that the elements of each member of Φ form an interval. We can enlarge $\mathcal{A}$ by duplicating each of its elements ($\hat{\mathcal{A}} = \{a, a', b, b', c, c'\}$) and then we can take $\hat{\Phi} = \{\{a, b\}, \{a', c\}, \{b', c'\}\}$. Now

$$a\; b\; M_1 a'\; c\; M_2\; b'\; c'\; M_3 \quad (4)$$

is an S-alphabet of $(\hat{\mathcal{A}}, \hat{\Phi})$ and therefore an enlarged S-alphabet of $(\mathcal{A}, \Phi)$.[7]

Because we always find a finite, enlarged S-alphabet of $(\mathcal{A}, \Phi)$, a minimally enlarged S-alphabet exists.

DEFINITION 2.5. *An enlarged S-alphabet* $(\hat{\mathcal{A}}, \hat{\Sigma}, \hat{<})$ *of* $(\mathcal{A}, \Phi)$ *is said to be* optimal *iff it fulfills the following conditions: First, there exists no other enlarged*

S-alphabet $(\tilde{\mathcal{A}}, \tilde{\Sigma}, \tilde{\precsim})$ of $(\mathcal{A}, \Phi)$, *the alphabet* $\tilde{\mathcal{A}}$ *of which has fewer elements than* $\hat{\mathcal{A}}$ *and second, no other enlarged S-alphabet of* $(\mathcal{A}, \Phi)$ *has the same number of object elements, but fewer markers than* $(\hat{\mathcal{A}}, \hat{\Sigma}, \hat{\precsim})$.

Hence, an optimal S-alphabet has a minimal number of duplicated elements and as few markers as possible. However, optimal S-alphabets do not necessarily minimize the overall length of the S-alphabet, as the following example illustrates: The system of sets

$$\Phi = \{\{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, e\}, \{a, e, f\}, \{a, e, f, g\}\} \tag{5}$$

is S-encodable, and a good corresponding S-alphabet is

$$gfeaM_1bM_2cM_3dM_4 \tag{6}$$

Since no element in (6) is duplicated, (6) even forms an optimal S-alphabet. However, the overall length of (6) is not minimal, as the enlarged S-alphabet (7) shows:

$$gfeaM_1dcba'M_2 \tag{7}$$

However, due to the duplication of a, (7) is not an optimal enlarged S-alphabet of (5).

After this introduction of basic concepts the following questions will be investigated in the present paper:

1. Do Pāṇini's *Śivasûtras* form an optimal S-alphabet?
   Here we have to answer the questions:
   (a) Is the duplication of a phonological segment necessary?
   (b) If a duplicated element is necessary, does the duplication of the "h" lead to an S-alphabet with as few markers as possible?
2. Is there a general method to decide whether a set of sets is S-encodable?
3. If a system of sets is S-encodable, is there a systematical method to construct a good corresponding S-alphabet?

Kiparsky (1991) discusses question 1 and shows that its affirmation follows from the principle of economy and the logic of the special case and the general case used in the construction of Pāṇini's whole grammar.[8] I will answer all three questions, using a different approach. Furthermore, I will give some hints about how to find an enlargement of a set of sets leading to an optimal S-alphabet.

3. The Existence of *Śivasūtra*-Style Representations of Systems of Sets

3.1. A Brief Introduction to the Theory of Planar Graphs

Throughout this paper we will need some basic knowledge about planar graphs, which will be briefly introduced in this section.[9] A *graph* G is a pair (V, E) consisting of a set of *vertices* V and a set of *edges* $E \subseteq V \times V$. *Directed graphs* are graphs the edges of which are directed.

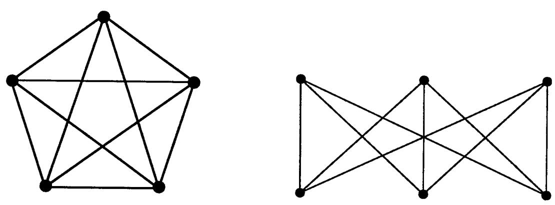
A graph is a *plane graph* if its vertices are points in the Euclidean plane $\mathbf{R} \times \mathbf{R}$ and its edges are polygonal arcs in $\mathbf{R} \times \mathbf{R}$, such that neither a vertex nor a point of an edge lies in the inner part of another edge. The Euclidean plane $\mathbf{R} \times \mathbf{R}$ is subdivided by a plane graph into *faces* (areas). Exactly one of these faces, the *infinite face,* is of unlimited size. Each edge of a plane graph lies at the *boundary* of at least one face, but not more than two faces.

If a graph is isomorphic to a plane graph, it is said to be *planar.* One of the most important criteria for the planarity of graphs is the criterion of Kuratowski, which is based on the notion of minors of a graph. A graph M is said to be a *minor* of a graph G if it can be constructed from G by first removing a number of vertices and edges from G and then contracting some of the remaining edges.

PROPOSITION 3.1. (Criterion of Kuratowski). *A graph G is planar iff G contains neither K^5 nor $K_{3,3}$ as a minor (see Figure 1).*

3.2. Plane Hasse-Diagrams and S-Encodability

Let $(\mathcal{A}, \Phi)$ be a system of sets as above, and let $\mathcal{H}(\Phi)$ be the set of all intersections of elements of $\Phi \cup \{\mathcal{A}\}$. $\mathcal{H}(\Phi)$ is partially ordered by the superset relation. A Hasse-diagram of a partially ordered set is a drawing of the directed graph whose vertices are the elements of the set and whose edges correspond to the upper neighbor relation determined by the partial order. The drawing must meet the following condition: if an element x of the partially ordered set is an upper neighbor of an element y, $y \prec x$, then the vertex of x lies above the vertex of y. In this paper we will stipulate that all edges are directed upwards.



a. K^5 b. $K_{3,3}$

Figure 1. The complete graph with 5 vertices, K^5, and the complete bipartite graph with $2 \cdot 3$ vertices, $K_{3,3}$.

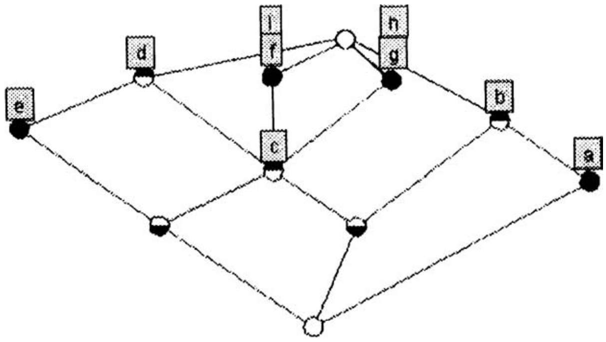


Figure 2. Hasse diagram of $(\mathcal{H}(\Phi), \supseteq)$, $\Phi = \{\{d, e\}, \{b, c, d, f, g, h, i\}, \{a, b\}, \{f, i\}, \{c, d, e, f, g, h, i\}, \{g, h\}\}$.

Figure 2 shows a drawing of the Hasse-diagram $(\mathcal{H}(\Phi), \supseteq)$ of our example system of sets $(\mathcal{A}, \Phi)$ from (1). All Hasse-diagrams in this paper are labeled economically as follows: for every $b \in \mathcal{A}$ the smallest set of $\mathcal{H}(\Phi)$ containing b (its b-set) is labeled. The set corresponding to a vertex of the diagram can be reconstructed by collecting all elements that are labeled to the vertex itself or to a vertex above. For example, in Figure 2 the vertex labeled with c corresponds to the set $\{c, d, f, g, h, i\}$.

The Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ gives a first hint for the question whether $(\mathcal{A}, \Phi)$ is S-encodable:

PROPOSITION 3.2. *If $(\mathcal{A}, \Phi)$ is S-encodable, then the Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ is a planar graph.*

Proof. The proof is based on the construction of a plane *stairs graph* of $(\mathcal{H}(\Phi), \subseteq)$.[10] Rotating the stairs graph of $(\mathcal{H}(\Phi), \subseteq)$ by 180° results in a plane Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$.[11]

Let $(\mathcal{A}, \Sigma, <)$ be an S-alphabet of $(\mathcal{A}, \Phi)$. For each element ϕ of $\mathcal{H}(\Phi)$ the coordinates in $\mathbf{R}^2$ of the corresponding vertex of the stairs graph are given as follows: The smallest element of ϕ w.r.t. $(\mathcal{A}, \Sigma, <)$ determines the x-coordinate of the vertex[12] and its y-coordinate is given by the length of the longest descending chain between ϕ and the empty set in $(\mathcal{H}(\Phi), \subseteq)$.[13]

The edges of the stairs graph are stair-shaped polygonal arcs: Let ϕ and ψ be two elements of $(\mathcal{H}(\Phi), \subseteq)$ with $\phi \prec \psi$. Since $\phi \prec \psi$ implies $\phi \subset \psi$ it follows for the x-coordinates that $\phi_x \geq \psi_x$. If $\phi_x = \psi_x$, then the edge between ϕ and ψ in the stairs graph is a straight line; otherwise the vertices ϕ and ψ are connected by the polygonal arc (see Figure 3)

$$(\psi_x, \psi_y), \left(\psi_x, \psi_y - \frac{1}{2}\right), \left(\phi_x - \frac{1}{2}, \psi_y - \frac{1}{2}\right), \left(\phi_x - \frac{1}{2}, \phi_y\right), (\phi_x, \phi_y).$$

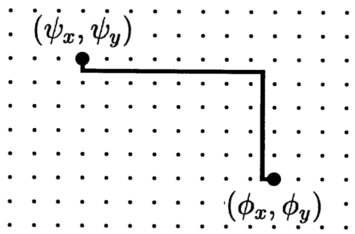


Figure 3. Edge of a stairs graph.

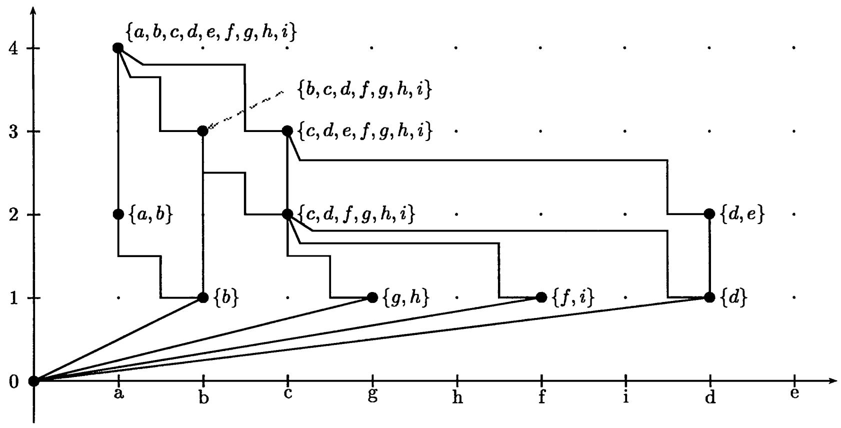


Figure 4. Stairs graph of $(\mathcal{H}(\Phi), \subseteq)$ with Φ={{*d, e*}, {*b, c, d, f, g, h, i*}, {a, b} {*f, i*}, {*c, d, e, f, g, h, i*}, {*g, h*}} constructed w.r.t. $abM_1cghM_2fiM_3dM_4eM_5$.

The only exception to this edge-construction rule is that every edge between a set ϕ and the empty set is just a straight line. The construction of the edges guarantees that no vertex of the stairs graph lies in the inner part of an edge.

With a simple but detailed case distinction it can be proved that every conflict that occurs between two edges (i.e. every intersection of two edges) can be solved by slightly transforming one of the edges in such a way that the distance between the transformed and the original edge does not exceed 1/4.[14] □

Figure 4 shows the stairs graph of $(\mathcal{H}(\Phi), \subseteq)$ with Φ taken from example (1); the stairs graph is constructed w.r.t. the S-alphabet given in (2).

As a corollary of Proposition 3.2 it follows that a system of sets is not S-encodable whenever the Hasse-diagram of the corresponding set of intersections is not planar.

Together with Kuratowski's Criterion 3.1 this answers question la, since Figure 5 shows a section of the Hasse-diagram of the natural classes of Sanskrit and their intersections, which has K^5 as a minor.[15] Hence, Pāṇini was forced to duplicate at least one of the phonological segments. But it remains to prove that h is the best candidate for the duplication; this discussion will be postponed.

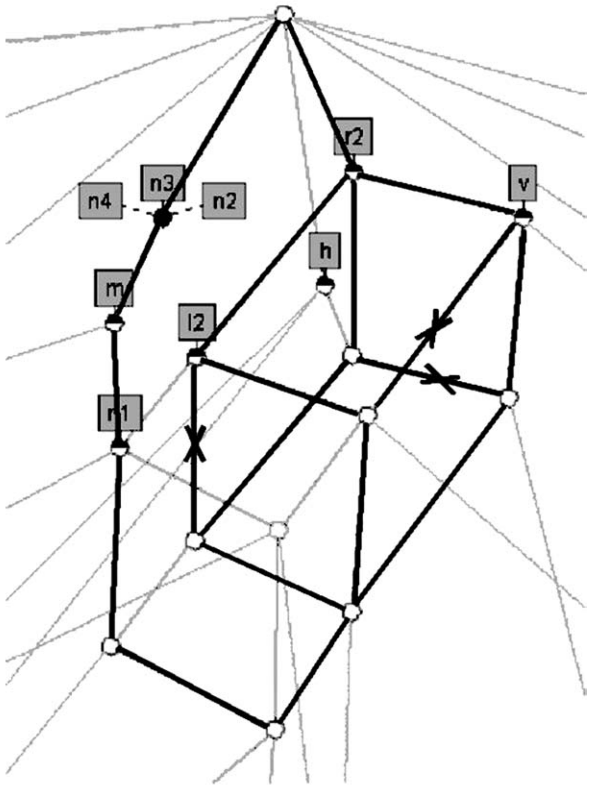


Figure 5. A section of the Hasse-diagram of the *pratyāhāras* used in the *Aṣṭādhyāyī* which has K^5 as a minor. The figure shows that the class memberships of the phonological segments *h,v* and *l* (denoted by *l*2) are independent of each other.

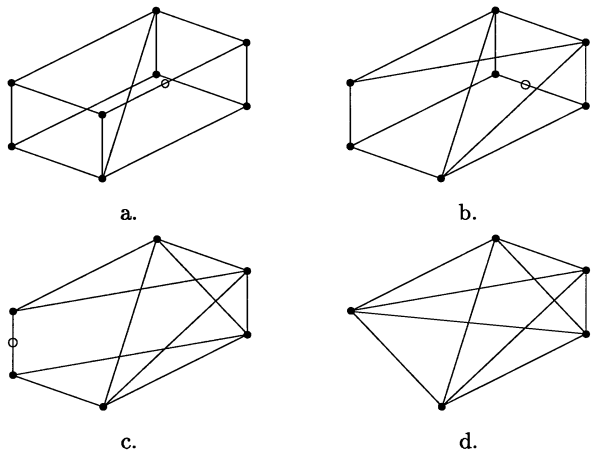


Figure 6. This sequence of graphs shows that the emphasized graph of Figure 5 has K^5 as a minor. The small circles indicate the edges which will be contracted in the next step.

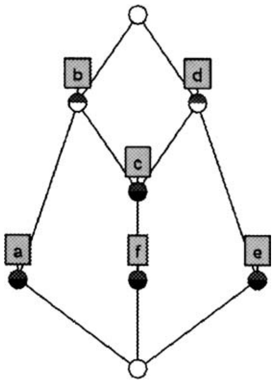


Figure 7. Hasse-diagram of the intersections of the sets $|\{d, e\}, \{a, b\}, \{b, c, d\}, \{b, c, d, f\}, \{a, b, c, d, e, f\}|$. It is plane, but there exists no corresponding S-alphabet.

PROPOSITION 3.3. *Pāṇini's pratyāhāras are not S-encodable.*

3.3. Boundary Graphs and S-Encodability

The condition for S-encodable systems of sets given in Proposition 3.2 is necessary but not sufficient, however. Figure 7 shows an example of a system of sets that is not S-encodable, although the Hasse-diagram of its intersection sets is planar. We need a stronger condition to fully identify those systems of sets which are S-encodable and to answer question 2.

THEOREM 3.4. *Let $(\mathcal{A}, \Phi)$ be a system of sets and $\bar{\Phi} = \Phi \cup \{\{a\} : a \in \mathcal{A}\}$. The following statements are equivalent:*

1. *$(\mathcal{A}, \Phi)$ is S-encodable.*
2. *The Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$ is planar.*
3. *If G is a plane Hasse-diagram of $(\mathcal{H}(\bar{\Phi})\backslash\emptyset, \supseteq)$, then for every $b \in \mathcal{A}$ the smallest set of $\mathcal{H}(\Phi)$ containing b (its b-set) is a vertex of the boundary $\Delta(G)$ of the infinite face of G. $\Delta(G)$ is called the boundary graph of $(\mathcal{H}(\Phi), \supseteq)$.*

Proof. By adding a new singleton $\{a\}$, $a \in \mathcal{A}$, to Φ, the S-encodability is preserved (at most one new marker immediately following a has to be inserted in an S-alphabet of $(\mathcal{A}, \Phi)$). Hence, $(\mathcal{A}, \Phi)$ is S-encodable iff $(\mathcal{A}, \bar{\Phi})$ is S-encodable. Together with Proposition 3.2, this proves that statement (1) implies statement (2).

Now we prove the equivalence of the statements (2) and (3). Adding a new set $\{a\}$ to $\mathcal{H}(\Phi)$ does not alter the Hasse-diagram much: a new vertex is created for $\{a\}$ and two new edges, one between $\{a\}$ and the a-set and the other between $\{a\}$ and the empty set. In a plane drawing of the Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$,

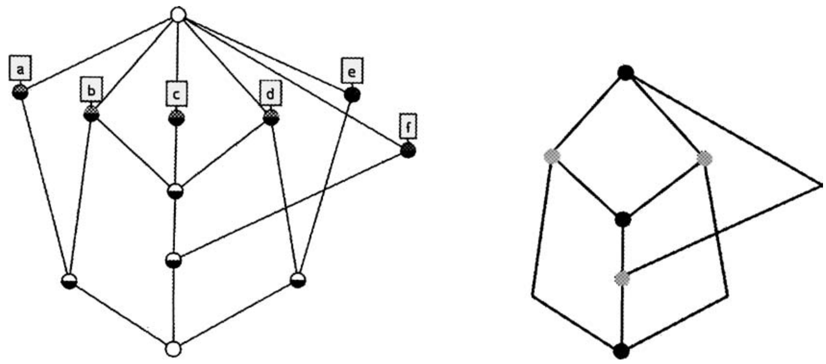


Figure 8. Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$ with $\bar{\Phi} = \{\{d, e\}, \{a, b\}, \{b, c, d\}, \{b, c, d, f\}, \{a, b, c, d, e, f\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ which has $K_{3,3}$ as a minor.

each singleton $\{a\}$ forms a vertex of the boundary graph; and if $\{a\} \notin \mathcal{H}(\Phi)$, then $\{a\}$ has exactly one lower neighbor in $(\mathcal{H}(\bar{\Phi}), \supseteq)$, namely the a-set of $\mathcal{H}(\Phi)$. Consequently, each a-set is an element of the corresponding boundary graph of $(\mathcal{H}(\Phi), \supseteq)$ if $(\mathcal{H}(\bar{\Phi}), \supseteq)$ is planar. And vice versa, if each a-set is an element of a boundary graph of $(\mathcal{H}(\bar{\Phi}), \supseteq)$, then we can add the extra vertices and edges of $(\mathcal{H}(\bar{\Phi}), \supseteq)$ without destroying the planarity.

It remains to prove that statement (2) implies statement (1). We first introduce the notion of a *run* through a boundary graph. Let $(\mathcal{A}, \Phi)$ be a system of sets such that the Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$ is planar. Furthermore, let G be a plane drawing of $(\mathcal{H}(\bar{\Phi})\backslash\emptyset, \supseteq)$ with boundary graph $\Delta(G)$. A *run* R through $\Delta(G)$ is a path that starts and ends at the vertex $\mathcal{A}$ and meets the following conditions: First, for every $a \in \mathcal{A}$ the path passes the a-set at least once; second, none of the edges occurring more than once in the path is part of a circle in $\Delta(G)$. Each run in $\Delta(G)$ induces a total order $<_R$ on $\mathcal{A}$: $a <_R b$ iff the run has passed the a-set at least once before it passes the b-set for the first time. The fact that G is a plane drawing of the Hasse-diagram of $(\mathcal{H}(\bar{\Phi})\backslash\emptyset, \supseteq)$ guarantees that each element of $\bar{\Phi}$ can be represented as an interval of $(\mathcal{A},<_R)$. Hence, $(\mathcal{A},<_R)$ can be extended to an S-alphabet of $(\mathcal{A}, \bar{\Phi})$ by putting a marker behind every element of $\mathcal{A}$. This concludes the proof of Theorem 3.4. □

Looking back at the example given in Figure 7, it is clear that by moving from Φ to $\bar{\Phi}$ the Hasse-diagram loses the quality of being planar; the left side of Figure 8 shows the Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$, which has the bipartite graph $K_{3,3}$ as a minor (drawing on the right side). Hence, $(\mathcal{A}, \Phi)$ cannot be S-encodable.

4. The Construction of *Śivasūtra*-Style Representations of Systems of Sets

4.1. The Boundary Graph Determines Good S-Alphabets

In the proof of Theorem 3.4 we presented a general method for the construction of an S-alphabet for every S-encodable system of sets $(\mathcal{A}, \Phi)$. We started from a run through a boundary graph of the Hasse-diagram of $(\mathcal{H}(\bar{\Phi}), \supseteq)$. As the resulting S-alphabets have as many markers as there are elements in $\mathcal{A}$, they generally are not good S-alphabets. The aim of this section is to develop a method for the construction of good S-alphabets.

Given an S-encodable system of sets $(\mathcal{A}, \Phi)$, the construction of the stairs graph in the proof of Proposition 3.2 presents a method for constructing a plane Hasse-diagram of $(\mathcal{H}(\Phi), \supseteq)$ for any S-alphabet of $(\mathcal{A}, \bar{\Phi})$ such that a run through its boundary graph exists which induces exactly the same S-alphabet. It follows that any total order of $\mathcal{A}$ given by an S-alphabet of $(\mathcal{A}, \Phi)$ can be induced by a run through a boundary graph of $(\mathcal{H}(\bar{\Phi}), \supseteq)$, too. If $(\mathcal{A}, \Phi)$ is an S-encodable system of sets, the boundary graph of $(\mathcal{H}(\Phi), \supseteq)$ is fixed up to isomorphism. This can be proved by induction over the length of the longest ascending chain in $(\mathcal{H}(\Phi), \supseteq)$ between a vertex and the vertex corresponding to the empty set.

To construct a good S-alphabet, we have to consider runs through the boundary graph of $(\mathcal{H}(\Phi), \supseteq)$, instead of $(\mathcal{H}(\bar{\Phi}), \supseteq)$. By looking at the boundary graph as a subgraph of the directed Hasse-diagram, the edges of a run can be directed.

An S-alphabet, seen as a sequence of markers and elements of $\mathcal{A}$, can be constructed from the empty sequence by traversing a run through the boundary graph of $(\mathcal{H}(\Phi), \supseteq)$ from the beginning to the end: If a vertex is reached which is labeled with an a-set, then one adds a to the sequence, together with all other labels of the same vertex. If an edge is passed whose direction contradicts the traversal direction, a new, previously unused, marker element is added to the sequence, unless the last added element is already a marker. Finally, after the end of the running path is reached, the sequence is revised as follows: If an element of $\mathcal{A}$ appears more than once in the sequence, delete all but the first occurrence. The definition of the boundary graph guarantees that, if a run passes an element $a \in \mathcal{A}$ more than once, the run goes upwards immediately after it reaches the a-set for the first time. Hence, eliminating all but the first occurrence of a reduces the number of markers in the resulting S-alphabet.

Applied to our small example (1) and the plane graph of its Hasse-diagram given in Figure 2, we may choose the run through the boundary graph illustrated in Figure 9. Traversing the running path, we pass first the a-set and the b-set without using an edge against its destined direction. Now we move downwards and violate the direction of the edge, and therefore we have to add a marker to our sequence, so that it starts with $a\ b\ M_1$. Now moving upwards we collect the c and the g, but since the g- and the h-sets are identical we also have to collect the h. After this we move downwards again, and that is why we add a new marker. We again reach the c-set and add c a second time to our sequence. So far our sequence is $a\ b\ M_1\ c\ g\ h\ M_2\ c$, and if we continue we end up with the S-alphabet depicted in (2).

Note that this procedure does not yield a unique S-alphabet because we have several decisions to make: (a) If a vertex is labeled with more than one element, their order in the S-alphabet is arbitrary; (b) from the vertex labeled c we can either go to the vertex labeled gh or if; (c) the path can be traversed clockwise or anti-clockwise.

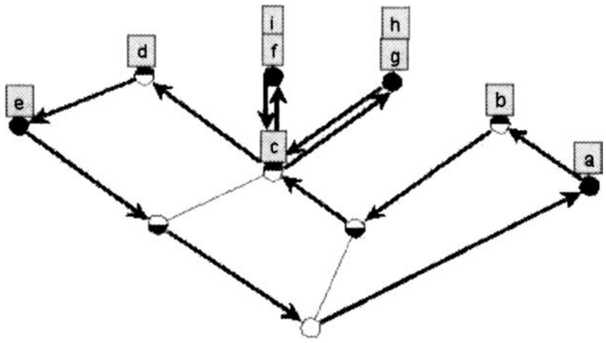


Figure 9. Hasse-diagram of example (1) with a possible path in its S-graph from which the S-alphabet $abM_1cghM_2fiM_3dM_4eM_5$ can be achieved.

Whenever a run violates the direction of an edge immediately after passing an a-set, a new marker has to be added to the S-alphabet. Hence, every good S-alphabet of $(\mathcal{A}, \Phi)$ can be constructed by finding a run through the boundary graph which minimizes the number of such marker-insertion situations. This answers question 3.

PROPOSITION 4.1. *If* $(\mathcal{A}, \Phi)$ *is S-encodable, then all good S-alphabets* $(\mathcal{A}, \Sigma, <)$ *of* $(\mathcal{A}, \Phi)$ *can be constructed systematically from the boundary graph of a plane Hasse-diagram of* $(\mathcal{H}(\Phi), \supseteq)$.

4.2. Pāṇini's Śivasūtras Are Optimal

Figure 10 shows the Hasse-diagram, with duplicated h, corresponding to the *pratyāhāras* which Pāṇini uses in his *Aṣṭādhyāyī*; the duplication of h is denoted by $h\_$.[16] The 42 *pratyāhāras* actually used by Pāṇini in the *Aṣṭādhyāyī* are marked in the figure with white boxes. The black and the striped rectangles next to some of the vertices mark the places where markers have to be added depending on the traversal direction (black: anti-clockwise [14 markers], striped: clockwise [17 markers]). It is obvious that no S-encoding can have less than 14 markers and the good S-alphabets are the various combinatorial variants of

$$\begin{aligned}&\langle \text{a, i, u}, M_1, \text{ṛ, ḷ}, M_2, \{\langle\{\text{e, o}\}, M_3\rangle, \langle\{\text{ai, au}\}M_4\rangle, \text{h, y, v, r}, M_5, \text{l}, M_6, \text{ñ, m},\\ &\quad \{\text{n̄, ṇ, n}, \}M_7, \text{jh, bh}, M_8, \{\text{gh, ḍh, dh}\}, M_9, \text{j}, \{\text{b, g, ḍ, d}\}, M_{10},\\ &\qquad \{\text{kh, ph}\}, \{\text{ch, ṭh, th}\}, \{\text{c, ṭ, t}\}, M_{11}, \{\text{k, p}\}, M_{12}, \{\text{s, ṣ, s}\}, M_{13}, \text{h}, M_{14}\rangle.\end{aligned}$$

Kiparsky (1991) argues in detail that the order chosen by Pāṇini out of the set of possibilities is unique if one requires a subsidiary principle of restrictiveness.

So far we have answered question 1a and questions 2 and 3 in the affirmative. Hence, we have argued that Pāṇini was forced to enlarge the alphabet, but it remains

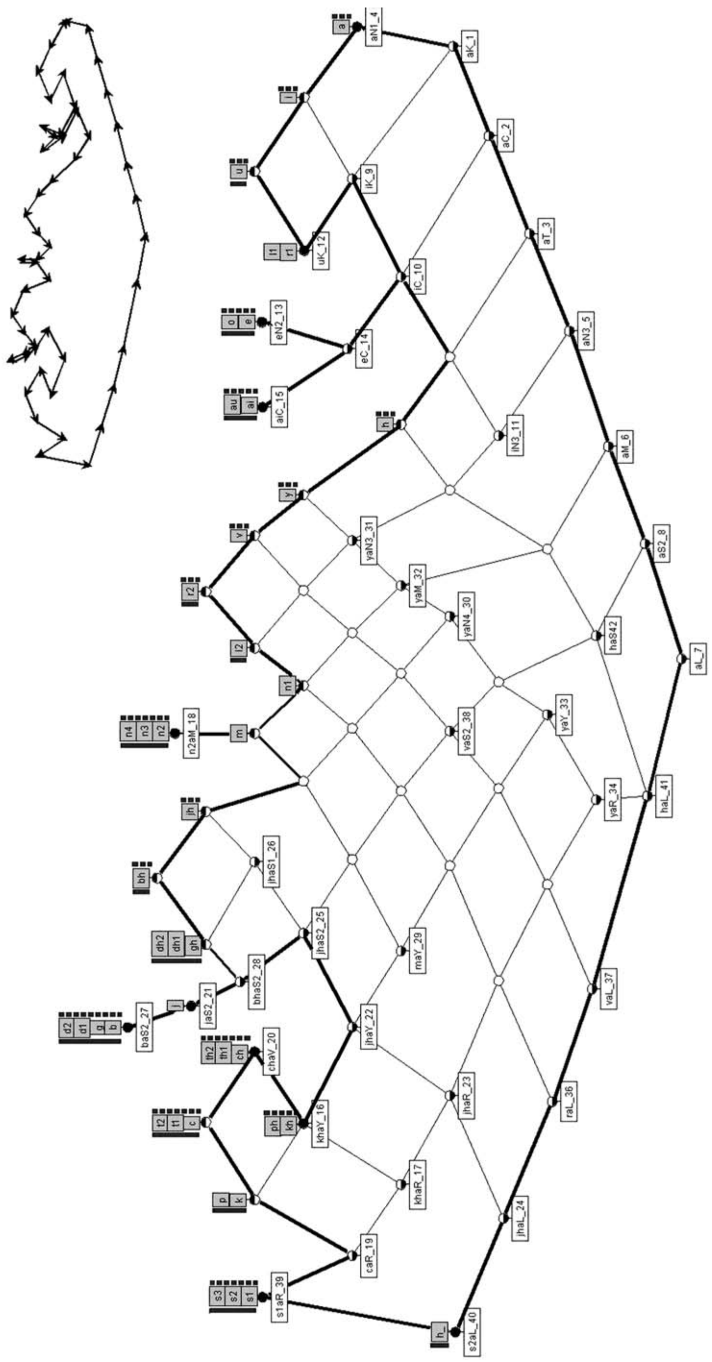


Figure 10. Hasse-diagram of the *pratyāhāras* used in the *Aṣṭādhyāyī*. The denotations in the figure are as follows: *h_* is the duplicate of *h* and r1:ṛ, l1:ḷ, r2:r, l2:l, n1:ñ, n2:ṅ, n3:ṇ, n4:n, dh1:ḍh, dh2:dh, d1:ḍ, d2:d, th1:ṭh, th2:th, t1:ṭ, t2:t, s1:ś, s2:ṣ, s3:s. The white boxes mark the 42 phonological classes which Pāṇini uses in the *Aṣṭādhyāyī*. The small figure on top shows the path in the S-graph which one has to choose to construct Pāṇini's *Śivasūtras*.

to show why duplicating the h is the best choice. If h is entirely removed from the natural classes, then a good S-alphabet has only one marker less, namely 13.

In the natural classes, the occurrences of h, v and l are independent of each other; since there exists a natural class – or an intersection of natural classes – for each subset of $\{h,l,v\}$ which contains the elements of the subset but no element of its complement in $\{h,l,v\}$. Therefore, the class memberships of the segments h,l and v are independent of each other. A Hasse-diagram that contains three independent elements has K^5 as a minor and is therefore not planar (see Figures 5 and 6). Triples of three independent elements are called K^5-*triples*. Hence, to get a planar Hasse-diagram it is necessary to duplicate at least one element of each K^5-*triple*.

Looking at the *pratyāhāras* used in the *Aṣṭādhyāyī* we find 249 K^5-*triples*; each of them contains h, and no other element is contained in each of them. Hence, to avoid the duplication of h it would be necessary to duplicate more than one element. For this reason there is no other choice than duplicating h to get an optimal S-alphabet corresponding to Pāṇini's *pratyāhāras*.

PROPOSITION 4.2. *Pāṇini's Śivasūtras form an optimal S-alphabet.*

Summarizing, all three questions at issue must be affirmed. Pāṇini's method of representing hierarchical information in a linear form is an interesting field of further investigations. Especially the fact that this method enables us to define phonological classes without referring to phonological features is remarkable.

Kornai (1993) points out that Pāṇini's approach is generalized by *feature geometry*, and that it is genuinely weaker than the latter. Although Kornai argues that the power of feature geometry is needed to get the proper set of natural classes of a phonological system, for some special tasks such as describing the set of *major class features* a *Śivasūtra*-style analysis can be appropriate.

However, after my investigation of several analyses of the phonological systems of different languages, Pāṇini's phonological theory of Sanskrit seems to have a special, rare property: The 249 K^5-*triples* of phonological elements created by the *pratyāhāras* used in the *Aṣṭādhyāyī* turn out to be a typical number of such triples, but the fact that each of these triples contains the element h is extraordinary. Table III shows an analysis of the natural classes of the consonants of German taken from Wurzel (1981). The 12 simple natural classes and their intersections yield 290 K^5-*triples*. But in any case, one needs to duplicate more than 8 of the 23 phonological elements to get a corresponding S-alphabet. A classification of vowels taken from Hall (2000) is given in Table IV. Here, although the number of K^5-*triples* is only 126 one needs to duplicate 7 elements in order to eliminate all K^5-*triples*. A resulting S-alphabet is, e.g., *εiɪyʏuʊɯ*M_1*oɔ a*M_2*æ*M_3*ɔʊʏœoφuy*M_4*ie*M_5. Up to now, the author only found the simple analysis of German vowels given in Table V, which succeeds with a single duplication in an optimal S-alphabet (*uüoö*M_1eM_2*iü*M_3*a*M_4).

Finally, it should be emphasized again: the approach presented is general and not limited to the domain of phonology.

Table III. Classification of the consonants of German, taken from Wurzel (1981).

	p	b	f	v	p̌	m	t	d	s	z	c	n	l	r	š	ç	J	k	g	x	ɣ	ŋ	R
[Nasal]						×						×										×	
[Obstruent]	×	×	×	×	×		×	×	×	×	×				×	×	×	×	×	×			
[Low]																							
[High]															×	×	×	×	×	×	×	×	
[Back]																		×	×	×	×	×	×
[Anterior]	×	×	×	×	×	×	×	×	×	×	×	×	×	×									
[Coronal]							×	×	×	×	×	×	×	×	×								
[Continuant]			×	×					×	×			×	×	×	×	×			×	×		×
[Fricative]			×	×	×				×	×	×		×	×	×	×	×			×	×		×
[Strident]			×	×	×				×	×	×				×	×	×			×	×		
[Voice]		×		×		×		×		×		×	×	×			×		×		×	×	×
[Lateral]													×										

Table IV. Classification of vowels, taken from Hall (2000). An optimal corresponding enlarged S-alphabet is **εiɪyʏuʊɯ**$\mathbf{M_1}$**oɔa**$\mathbf{M_2}$**æ**$\mathbf{M_3}$**ɔʊʏœoøuy**$\mathbf{M_4}$**ie**$\mathbf{M_5}$. The *pratyāhāra* of the class of back vowels is **u**$\mathbf{M_2}$, the high vowels are denoted by **i**$\mathbf{M_1}$ the low vowels by **a**$\mathbf{M_3}$, the tense vowels by **o**$\mathbf{M_5}$, and the round vowels by **ɔ**$\mathbf{M_4}$.

	i	ɪ	y	ʏ	e	ε	ɸ	œ	æ	u	ʊ	ɯ	o	ɔ	a
[Back]										×	×	×	×	×	×
[High]	×	×	×	×						×	×	×			
[Low]									×						×
[Tense]	×		×		×		×			×			×		
[Round]			×	×			×	×		×	×		×	×	

Table V. Example classification of vowels. An optimal corresponding enlarged S-alphabet is **uüoö**$\mathbf{M_1}$**e**$\mathbf{M_2}$**iü**$\mathbf{M_3}$**a**$\mathbf{M_4}$.

	a	i	u	ü	e	o	ö
[Front]		×		×	×		×
[Round]			×	×		×	×
[Mid]					×	×	×
[Low]	×						

Notes

1. This approach fits naturally in the framework of Formal Concept Analysis (see Ganter & Wille 1999), because the investigated graphs are formal concept lattices.
2. The *Śivasūtras* allow a total of 305 pairs to be constructed consisting of a phonological segment followed by an anubandha. Due to the double occurrence of 'h', 10 of the pairs denote the same set of phonological segments (e.g. 'aL' and 'aR'). If we exclude classes containing a single element, then the number of phonological classes which can be expressed by Pāṇini's *Śivasūtras* reduces to $305 - 10 - 14 = 281$.
3. Since Pāṇini's grammar was designed for oral tradition, the linear form of the *Śivasūtras* was a prerequisite.
4. The following consideration shows that one does not need to enumerate all elements of an interval to decide whether a special element belongs to this interval. If a trained person is looking for the word 'enzyme' and opens a lexicon by chance at the letter 'M', then it is clear that the entry sought is contained in the preceding part of the book.
5. Duplicating an element a means adding a new element a' to $\mathcal{A}$ and changing some of the occurrences of a in Φ to a'.
6. An S-alphabet of $(\hat{\mathcal{A}}, \hat{\Phi})$ will sometimes be called an *enlarged S-alphabet of* $(\mathcal{A}, \Phi)$.
7. This example only illustrates a general method of constructing an enlarged S-alphabet. It is clear that there are better enlargements, leading to more compact S-alphabets (e.g. $abM_1cM_2a'M_3$).
8. Another discussion of question 1 can be found in Staal (1962).
9. The full details can be found in every introductory book on graph theory. In writing this paper Diestel (1997) proved to be especially helpful.
10. Figure 4 shows an example of a stairs graph.
11. The concentration on the ordered set $(\mathcal{H}(\Phi), \supseteq)$ stems from the fact that the approach presented in this paper was originally developed in the framework of Formal Concept Analysis.

12. If the smallest element of ϕ is the n-th smallest element of $\mathcal{A}$ in $(A, \Sigma, <)$, then the x-coordinate is n. The x-coordinate of the empty set is 0.
13. Throughout the whole paper, we consider only the case that $\emptyset \in \mathcal{H}(\Phi)$. It can be easily shown that all propositions remain true in the simpler case $\emptyset \notin \mathcal{H}(\Phi)$.
14. Due to the limited space, proofs are not given in full detail here, but the results are illustrated by a number of examples.
15. The emphasized lines in the figure mark a way to arrive at the minor K^5: remove all edges which are not emphasized and contract those edges which are marked by arrows. The sequence of graphs in Figure 6 shows the result of each step of contraction.
16. The drawing was done by the tool "Concept Explorer" written by Sergey Yevtushenko (http://www.sourceforge.net/projects/conexp).

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