Mildly Context-Sensitive Grammar Formalisms:

Linear Context-Free Rewriting Languages: Formal Properties

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Overview

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Pumping Lemma: Intuition (1)

LCFRS have a context-free backbone: the productions constitute a *generalized context-free grammar*. A derivation step consists of replacing a lhs of a production with its rhs.

Example (LCFRS for the copy language):

$S \to f_1[A] \quad A \to f_2[A] \quad A \to f_3[A] \quad A \to f_4[\,] \quad A \to f_5[\,]$

$f_1[\langle X, Y, Z\rangle] = \langle XYZ\rangle \qquad f_4[\,] = \langle a, a, a\rangle$
$f_2[\langle X, Y, Z\rangle] = \langle aX, aY, aZ\rangle \qquad f_5[\,] = \langle b, b, b\rangle$
$f_3[\langle X, Y, Z\rangle] = \langle bX, bY, bZ\rangle$

Derivation in underlying generalized CFG:

$S \Rightarrow f_1(A) \Rightarrow f_1(f_3(A)) \Rightarrow f_1(f_3(f_2(A))) \Rightarrow f_1(f_3(f_2(f_4())))$

The term $f_1(f_3(f_2(f_4())))$ denotes $\langle baabaa\rangle$.

Pumping Lemma: Intuition (2)

- In such a derivation, the expansion of a non-terminal A does not depend on the context A occurs in.
- Consequently, as in the case of CFG, if we have a derivation

$$A \stackrel{+}{\Rightarrow} f_1(\dots f_2(\dots f_k(\dots A \dots) \dots) \dots)$$

then this part of the derivation can be iterated, i.e., we can also have

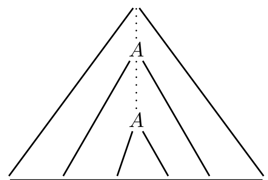
$$A \stackrel{+}{\Rightarrow} f_1(\dots f_2(\dots f_k(\dots f_1(\dots f_2(\dots f_k(\dots A \dots) \dots) \dots) \dots) \dots) \dots)$$

etc.

Pumping Lemma (1)

Question: What does this mean for the string language?

Assume that we have such an iteration, i.e., in the derivation tree, we have



with no other derivation $B \stackrel{+}{\Rightarrow} B$ in the subtree corresponding to $A \stackrel{+}{\Rightarrow} A$. The part between the two A nodes can be iterated.

Pumping Lemma (2)

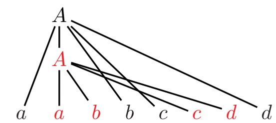
- Let m be the fan-out (the arity) of A. Then the higher A spans an m-tuple of strings and the lower A spans a (smaller) m-tuple of strings that is part of the m-tuple of the higher A. Assume that $\langle w_1, \dots, w_m \rangle$ is the span of the lower A.
- There are different cases for how the components of the lower A are part of the span of the higher A. Either w_i is part of the ith component $(1 \leq i \leq m)$ or there are components of the higher A that do not contain parts of the span of the lower A.

Pumping Lemma (3)

Case 1: w_i is part of the ith component of the higher A $(1 \leq i \leq m)$. Then the span of the higher A has the form $\langle v_1 w_1 u_1, \dots, v_m w_m u_m \rangle$.

Consequently (iteration), $\langle v_1^k w_1 u_1^k, \dots, v_m^k w_m u_m^k \rangle$ is also in the yield of A.

Example:
$S(XY) \rightarrow A(X,Y), A(aXb, cYd) \rightarrow A(X,Y), A(ab, cd) \rightarrow \varepsilon$



Iteration:
$a^n abb^n c^n cdd^n$

The iterated parts are present in the original string (in the tree with just two As on the path).

Pumping Lemma (4)

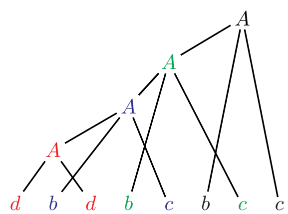
Case 2: The $w_1, \dots, w_m$ are part of only j components $(j < m)$ of the span of the higher A. Then, when iterating, the components of the higher A go again into only j components, i.e., the $m - j$ components that do not contain any of the $w_1, \dots, w_m$ must be added to the other ones.

Consequently, in a component of the higher A, we either have the form $v_1 w_i v_2 w_{i+1} \dots v_{k-1} w_{i+k} v_k$ or a form u (without components from the lower A).

In the next iteration, the u will be added to one of the other components. This can be repeated and will lead to iterations of strings that are concatenations of some of the u and some of the v_i. These iterated strings are not necessarily present in the span of the higher A, before iteration.

Pumping Lemma (5)

Example:
$S(XYZU) \to A(XYZ, U), A(XbY, c) \to A(X, Y), A(d, d) \to \varepsilon$



Iteration pattern: $dbd(bc)^n c$

Here, the iterated parts are not present in the original string (in the tree with just two As on the path).

Pumping Lemma (6)

Along these lines, [Seki et al., 1991] show the following pumping lemma for k-MCFLs, the class of languages generated by k-MCFGs:

Proposition 1 (Pumping Lemma for k-MCFLs) *For any k-MCFL L, if L is an infinite set then there exist some $u_j \in T^*$ $(1 \leq j \leq k+1)$, $v_j, w_j, s_j \in T^* (1 \leq j \leq k)$ which satisfy the following conditions:*

1. $\Sigma_{j=1}^{k} |v_j s_j| > 0$, *and*
2. *for any* $i \geq 0$,

$$z_i = u_1 v_1^i w_1 s_1^i u_2 v_2^i w_2 s_2^i \dots u_k v_k^i w_k s_k^i u_{k+1} \in L$$

Pumping Lemma (7)

- Note that the pumping lemma is only *existential* in the sense that it does not say that within each string that is long enough we find pumpable substrings.
- It only says that there exist strings in the language that are of a limited length and that contain pumpable substrings.
- In contrast to this, the CFG pumping lemma is *universal*: within every string of sufficient length we find two pumpable substrings of a limited distance.

Pumping Lemma: Applications (1)

Proposition 2 *For every $k \geq 1$, the language $\{a_1^n a_2^n \dots a_{2k+1}^n \mid n \geq 0\}$ is not a k-MCFL.*

Proof: Assume that it is a k-MCFL. Then it satisfies the pumping lemma with $2k$ pumpable strings. At least one of these strings is not empty and none of them can contain different terminals. However, if at most $2k$ strings are pumped, we necessarily obtain strings that are not in the language. Contradiction.

Pumping Lemma: Applications (2)

For every $k \geq 1$, the language $\{a_1^n a_2^n \dots a_{2k+1}^n \mid n \geq 0\}$ is a $(k+1)$-MCFL.

It is generated by an MCFG/LCFRS with the following rules:

$$S(X_1 X_2 \dots X_k X_{k+1}) \to A(X_1, X_2, \dots, X_k, X_{k+1})$$
$$A(a_1 X_1 a_2, a_3 X_2 a_4, \dots, a_{2k-1} X_k a_{2k}, a_{2k+1} X_{k+1}) \to A(X_1, X_2, \dots, X_k, X_{k+1})$$
$$A(\varepsilon, \dots, \varepsilon) \to \varepsilon$$

Proposition 3 *k-MCFL is a proper subset of $(k+1)$-MCFL.*

Closure Properties

[Seki et al., 1991] show the following closure properties for k-MCFL:

Proposition 4 *For every $k \geq 1$, the class k-MCFL*

- *is closed under substitution;*
- *is closed under union, concatenation, Kleene closure, ε-free Kleene closure;*
- *is closed under intersection with regular languages.*

Closure Properties: Substitution

k-MCFL being closed under substitution means:

If L is a k-MCFL over the terminal alphabet T and f assigns a k-MCFL to every $t \in T$, then

$$f(L) = \{w_1 \dots w_n \mid \text{ there is a } t_1 \dots t_n \in L \text{ with } w_i \in f(t_i) \text{ for } 1 \leq i \leq n\}$$

is also a k-MCFL.

Idea of the construction of the new k-MCFG from the original one and the ones for the images of the terminals: take the original and replace every terminal a in a lhs with a new variable X_a and add $S_a(X_a)$ to the rhs where S_a the start symbol of the grammar of the image of a.

Closure Properties: Union and Concatenation

Let L_1, L_2 be languages generated by the k-MCFGs G_1, G_2 with start symbols S_1, S_2 respectively (and without loss of generality disjoint sets of non-terminals).

- The union, $L_1 \cup L_2$ is generated by the grammar with the rules from G_1 and G_2 and additional rules $S(X) \to S_1(X)$, $S(X) \to S_2(X)$ where S is a new start symbol.
- The concatenation $\{w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2\}$ is generated by the grammar with the rules from G_1 and G_2 and an additional rule $S(XY) \to S_1(X) S_2(Y)$ where S is a new start symbol.

Closure Properties: Kleene star

Let L be a language generated by the k-MCFGs G.

- If we add the rules $S'(XY) \to S(X)S'(Y)$ and $S'(\varepsilon) \to \varepsilon$ where S' is a new start symbol, we generate the Kleene closure L^* of L.
- If we add the rules $S'(XY) \to S(X)S'(Y)$ and $S'(X) \to S(X)$ where S' is a new start symbol, we generate the ε-free Kleene closure L^+ of L.

Closure Properties: Intersection with regular lang. (1)

Construction idea: enrich the non-terminals A with lists of states $q_1, q_1', \dots, q_{dim(A)}, q'_{dim(A)}$ where the path from q_i to q_i' is the path traversed while processing the ith component of A.

Example:

Take the copy language, generated by an MCFG with

$S(XY) \to A(X,Y)$

$A(aX, aY) \to A(X,Y) \quad A(bX, bY) \to A(X,Y) \quad A(\varepsilon, \varepsilon) \to \varepsilon$

Intersect with $a^*b^*a^*b^*$, generated by a DFA with $Q = F = \{q_0, q_1, q_2, q_3\}$, initial state q_0 and

$\delta(q_0, a) = q_0, \delta(q_0, b) = q_1, \delta(q_1, b) = q_1, \delta(q_1, a) = q_2,$

$\delta(q_2, a) = q_2, \delta(q_2, b) = q_3, \delta(q_3, b) = q_3.$

Closure Properties: Intersection with regular lang. (2)

Result:

a^* $S[q_0,q_0](XY) \to A[q_0,q_0,q_0,q_0](X,Y)$,
$A[q_0,q_0,q_0,q_0](aX,aY) \to A[q_0,q_0,q_0,q_0](X,Y)$,
$A[q_0,q_0,q_0,q_0](\varepsilon,\varepsilon) \to \varepsilon$

b^+ $S[q_0,q_1](XY) \to A[q_0,q_1,q_1,q_1](X,Y)$,
$A[q_0,q_1,q_1,q_1](bX,bY) \to A[q_1,q_1,q_1,q_1](X,Y)$,
$A[q_1,q_1,q_1,q_1](bX,bY) \to A[q_1,q_1,q_1,q_1](X,Y)$,
$A[q_1,q_1,q_1,q_1](\varepsilon,\varepsilon) \to \varepsilon$

$a^+b^+a^+b^+$ $S[q_0,q_3](XY) \to A[q_0,q_1,q_1,q_3](X,Y)$,
$A[q_0,q_1,q_1,q_3](aX,aY) \to A[q_0,q_1,q_2,q_3](X,Y)$,
$A[q_0,q_1,q_2,q_3](aX,aY) \to A[q_0,q_1,q_2,q_3](X,Y)$,
$A[q_0,q_1,q_2,q_3](bX,bY) \to A[q_1,q_1,q_3,q_3](X,Y)$,
$A[q_1,q_1,q_3,q_3](bX,bY) \to A[q_1,q_1,q_3,q_3](X,Y)$,
$A[q_1,q_1,q_3,q_3](\varepsilon,\varepsilon) \to \varepsilon$

Closure Properties: Intersection with regular lang. (3)

Proposition 5 *[Kallmeyer, 2010]*

$L = \{(a^m b^m)^n \mid m, n \geq 1\}$ *is not an MCFL.*

Proof: We assume that there is a fixed k such that there is a k-MCFG generating L.

We intersect L with the regular language $(a^+b^+)^{k+1}$, which yields $L' = \{(a^m b^m)^{k+1} \mid m \geq 1\}$. L' does not satisfy the pumping lemma for k-MCFL since the iterated parts in the pumping lemma must each consist of either as or bs (otherwise we would increase the number of substrings a^m and b^m when iterating). Furthermore, if we have at most $2k$ iterated parts, the iterations necessarily lead to words where the a^m and b^m parts no longer have all the same exponent. Consequently, L' and therefore also L are not k-MCFLs. Since this holds for any k, L is not an MCFL.

References

[Kallmeyer, 2010] Kallmeyer, L. (2010). *Parsing Beyond Context-Free Grammars*. Cognitive Technologies. Springer, Heidelberg.

[Seki et al., 1991] Seki, H., Matsumura, T., Fujii, M., and Kasami, T. (1991). On multiple context-free grammars. *Theoretical Computer Science*, 88(2):191–229.