Einführung in die Computerlinguistik

Context-Free Grammars (CFG)

Laura Kallmeyer

Heinrich-Heine-Universität Düsseldorf

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CFG (1)

In contrast to regular grammars, CFGs allow any combination of non-terminal and terminal symbols in the righthand sides of productions.

CFG $G_{telescope}$

- Nonterminals {S, NP, VP, PP, D, N, V, P}
- Terminals {man, girl, the, John, Mary, telescope, with, saw}
- Start symbol S
- Productions

S $\rightarrow$ NP VP	NP $\rightarrow$ D N	N $\rightarrow$ N PP
VP $\rightarrow$ VP PP	VP $\rightarrow$ V NP	PP $\rightarrow$ P NP
N $\rightarrow$ man	N $\rightarrow$ girl	N $\rightarrow$ telescope
D $\rightarrow$ the	NP $\rightarrow$ John	NP $\rightarrow$ Mary
P $\rightarrow$ with	V $\rightarrow$ saw	

CFG (2)

Example continued

$S \rightarrow NP\ VP$	$NP \rightarrow D\ N$	$N \rightarrow N\ PP$	$VP \rightarrow VP\ PP$
$VP \rightarrow V\ NP$	$PP \rightarrow P\ NP$	$N \rightarrow$ man	$N \rightarrow$ girl
$N \rightarrow$ telescope	$D \rightarrow$ the	$NP \rightarrow$ John	$NP \rightarrow$ Mary
$P \rightarrow$ with	$V \rightarrow$ saw		

Sentences one can generate with this grammar:

(1) John saw Mary

(2) John saw the girl

(3) the man with the telescope saw John

(4) John saw the girl with the telescope

…

CFG (3)

CFG

A context-free grammar (CFG) is a tuple $G = \langle N, T, P, S \rangle$ such that

- N and T are disjoint alphabets, the nonterminals and terminals,
- $S \in N$ is the start symbol, and
- P is a set of productions of the form

$$A \to \beta$$

  with $A \in N, \beta \in (N \cup T)^*$.

Any $\beta \in (N \cup T)^*$ with $S \overset{*}{\Rightarrow} \beta$ is called a sentential form.

CFG (4)

CFGs

- CFG $G_{anbn} = \langle \{S\}, \{a, b\}, P, S \rangle$ with productions

  $\mathrm{S} \rightarrow \mathrm{aSb} \quad \mathrm{S} \rightarrow \varepsilon$

  generates the languge $\{a^n b^n \mid n \geq 0\}$

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- CFG $G_{a,b} = \langle \{S, A, B\}, \{a, b\}, P, S \rangle$ with productions

  $$\begin{array}{lll} \mathrm{S} \rightarrow \mathrm{aB} & \mathrm{S} \rightarrow \mathrm{bA} & \\ \mathrm{A} \rightarrow \mathrm{a} & \mathrm{A} \rightarrow \mathrm{aS} & \mathrm{A} \rightarrow \mathrm{bAA} \\ \mathrm{B} \rightarrow \mathrm{b} & \mathrm{B} \rightarrow \mathrm{bS} & \mathrm{B} \rightarrow \mathrm{aBB} \end{array}$$

  generates the languge $\{w \mid w \in \{a, b\}^+, |w|_a = |w|_b\}$

CFG (5)

Parse tree

A tree t is a parse tree for a CFG $G = \langle N, T, P, S \rangle$ iff

- each node in t is labeled with a $x \in N \cup T \cup \{\epsilon\}$;
- the root label is S;
- if there is a node with label A that has n daughters labeled (from left to right) $x_1, \ldots, x_n$, then $A \rightarrow x_1 \ldots x_n \in P$.
- if a node has label ϵ, it is a leaf and the unique daughter of its mother node.

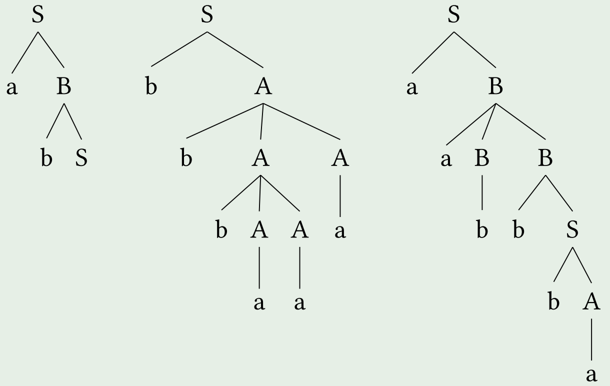
$S \stackrel{*}{\Rightarrow} \alpha$ in G iff there is a parse tree for G with yield α.

CFG (6)

Parse trees for $G_{a,b}$

$S \to aB \quad S \to bA \quad A \to a \quad A \to aS \quad A \to bAA$

$B \to b \quad B \to bS \quad B \to aBB$



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String language, derivation, tree language

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  - $\stackrel{*}{\Rightarrow}$ is the reflexive transitive closure of $\Rightarrow$.
- A derivation of a word $w \in T^*$ is a sequence $S \Rightarrow \alpha_1 \cdots \Rightarrow w$ of derivation steps leading to w.
- The tree language is the set of all parse trees with all leaves labelled with $a \in T \cup \{\varepsilon\}$.

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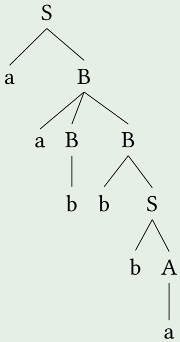
- leftmost derivation iff, in each derivation step, a production is applied to the leftmost non-terminal of the already derived sentential form.
- rightmost derivation iff, in each derivation step, a production is applied to the rightmost non-terminal of the already derived sentential form.

CFG (9)

Derivations

Take the grammar $G_{a,b}$ and the following parse tree:



Leftmost derivation:

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabB$
$\Rightarrow aabbS \Rightarrow aabbbA \Rightarrow aabbba$

Rightmost derivation:

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aaBbS$
$\Rightarrow aaBbbA \Rightarrow aaBbba \Rightarrow aabbba$

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  $G_{a,b}$ with $w = aabbab$

- A CFL L is called inherently ambiguous if each CFG G with $L = L(G)$ is ambiguous.

  Example:
  $\{a^n b^n c^m d^m \mid n \geq 1, m \geq 1\} \cup \{a^n b^m c^m d^n \mid n \geq 1, m \geq 1\}$

PDA (1)

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- the current state,
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Each move consists of

- changing state,
- popping the topmost symbol from the stack, and
- pushing a new sequence of symbols on the stack.

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Language $\{a^n c b^n \mid n \geq 0\}$

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CFLs are the languages accepted by (non-deterministic) PDAs.

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Equivalently, one can even define δ as
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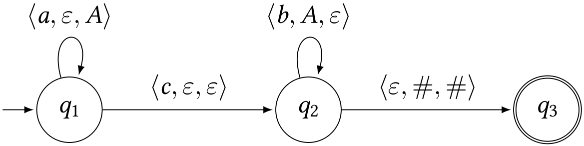
PDA (5)

PDAs can also be drawn as graphs:



An edge label $\langle a, A, B\rangle$ signifies that a is read, A is popped from the stack and B is pushed on the stack.

In other words, $\langle q_2, B\rangle \in \delta(q_1, a, A)$ iff there is an edge from q_1 to q_2 labeled $\langle a, A, B\rangle$.

PDA (6)

Instantaneous description

An instantaneous description of a PDA is a triple (q, w, γ) with

- $q \in Q$ is the current state of the automaton,
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$(q, aw, Z\alpha) \vdash (q', w, \beta\alpha)$ iff $\langle q', \beta \rangle \in \delta(q, a, Z)$ for all $q, q' \in Q$, $a \in \Sigma \cup \{\epsilon\}$, $w \in \Sigma^*$, $Z \in \Gamma$, $\alpha, \beta \in \Gamma^*$.

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$\overset{*}{\vdash}$ is the reflexive transitive closure of $\vdash$.

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- The language accepted by M with a final state is

  $L(M) := \{ w \mid (q_0, w, Z_0) \overset{*}{\vdash} (q_f, \epsilon, \gamma) \text{ for a } q_f \in F \text{ and a } \gamma \in \Gamma^* \}$

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The two modes of acceptance are equivalent, i.e., for each language L: there is a PDA M_1 with $L = L(M_1)$ iff there is a PDA M_2 with $L = N(M_2)$.

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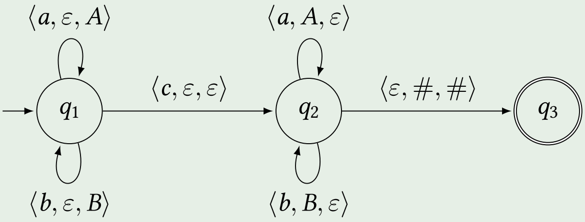
PDA M for $L(M) = \{wcw^R \mid w \in \{a, b\}^*\}$



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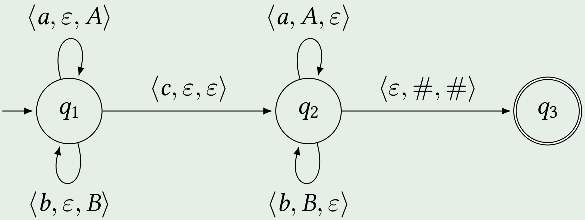

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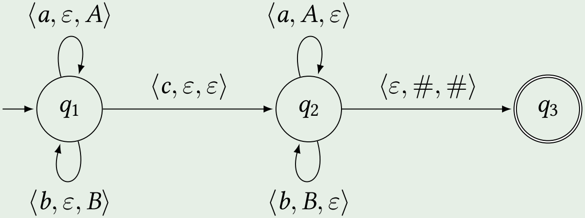

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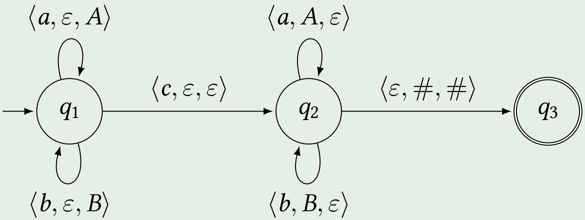

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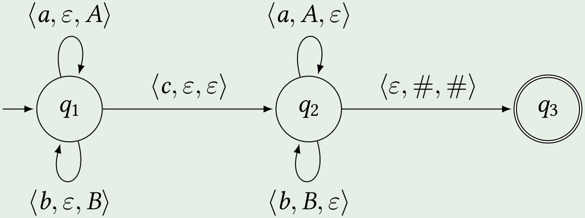

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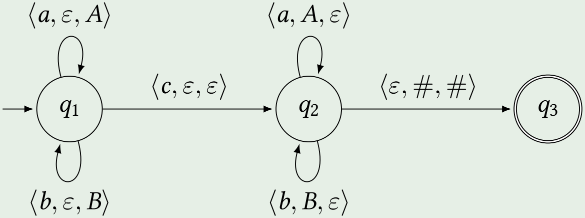

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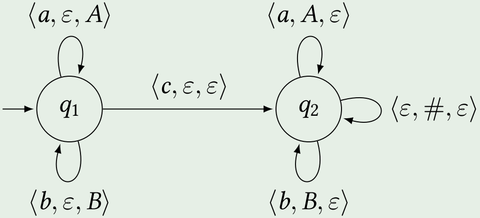

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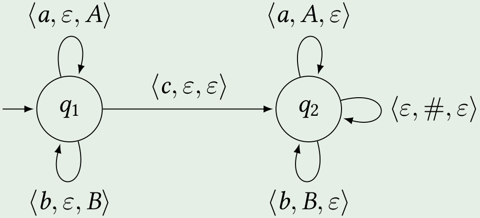

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- $\delta(q_1, c, \varepsilon) = \{\langle q_2, \varepsilon\rangle\}$
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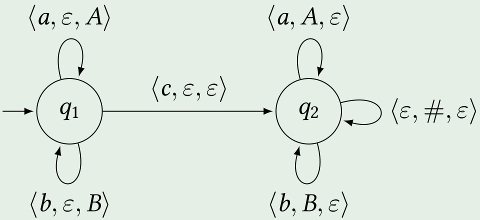
PDA (9)

PDA M for $N(M) = \{wcw^R \mid w \in \{a, b\}^*\}$



- $Q = \{q_1, q_2\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{\#, A, B\}$.
  $q_0 = q_1$, $Z_0 = \#$, $F = \emptyset$.
- $\delta(q_1, a, \varepsilon) = \{\langle q_1, A\rangle\}$ $\quad \delta(q_1, b, \varepsilon) = \{\langle q_1, B\rangle\}$
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- $\delta(q_2, \epsilon, \#) = \{\langle q_2, \epsilon\rangle\}$

PDA (10)

DPDA

A PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ is a deterministic PDA (DPDA) iff

- for all $q \in Q, Z \in \Gamma, a \in \Sigma \cup \{\epsilon\}$: $|\delta(q, a, Z)| \leq 1$, and
- for all $q \in Q, Z \in \Gamma$: if $\delta(q, \epsilon, Z) \neq \emptyset$, then $\delta(q, a, Z) = \emptyset$ for all $a \in \Sigma$.

The preceding example was a DPDA.

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The preceding example was a DPDA.

The class of languages accepted by DPDAs is smaller than the class accepted by (non-deterministic) PDAs.

Example of a language that requires a non-determinstic PDA: $\{ww^R \mid w \in \{a, b\}^*\}$.

PDA and CFG (1)

For each CFL L, there is a PDA M with $L = N(M)$.

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Construction: Assume that $\epsilon \notin L$. $L = L(G)$ for a CFG $G = \langle N, T, P, S \rangle$ in Greibach-Normal Form (GNF).
This means that all productions have the form $A \rightarrow a\gamma$ with $A \in N, a \in T, \gamma \in N^*$.
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The automaton simulates leftmost derivations in G.

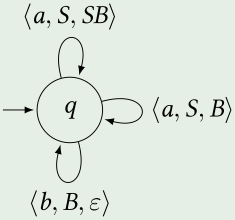
PDA and CFG (2)

From CFG to PDA

Take the CFG $G_{anbn} = \langle \{S\}, \{a, b\}, \{S \rightarrow aSb, S \rightarrow ab\}, S \rangle$.

Equivalent CFG in GNF:
$G'_{anbn} = \langle \{S, B\}, \{a, b\}, \{S \rightarrow aSB, S \rightarrow aB, B \rightarrow b\}, S \rangle$.

Equivalent PDA: $M = \langle \{q\}, \{a, b\}, \{S, B\}, \delta, q, S, \emptyset \rangle$ with acceptance with empty stack:



PDA and CFG (3)

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Construction of equivalent CFG for given PDA
$M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$:

- nonterminals: S and all $[q_1, Z, q_2]$ with $q_1, q_2 \in Q$, $Z \in \Gamma$.
- productions: $S \rightarrow [q_0, Z_0, q]$ for every $q \in Q$, and
  $[q, A, q_{m+1}] \rightarrow a[q_1, B_1, q_2], \ldots, [q_m, B_m, q_{m+1}]$ for $q, q_1, \ldots, q_{m+1} \in Q$, $a \in \Sigma \cup \{\epsilon\}$, $A, B_1, \ldots, B_m \in \Gamma$ such that $\langle q_1, B_1 \ldots B_m \rangle \in \delta(q, a, A)$
  $[q, A, q_1] \rightarrow a$ if $\langle q_1, \epsilon \rangle \in \delta(q, a, A)$.

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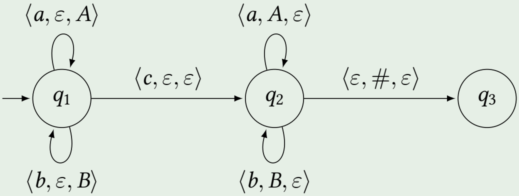
  $[q, A, q_1] \to a$ if $\langle q_1, \epsilon \rangle \in \delta(q, a, A)$.

$[q_1, A, q_2] \overset{*}{\Rightarrow} w$ iff $(q_1, w, A) \overset{*}{\vdash} (q_2, \epsilon, \epsilon)$.

PDA and CFG (4)

From PDA to CFG



Productions of the equivalent CFG (only useful productions listed):

$S \rightarrow [q_1, \#, q_3] \quad [q_1, \#, q_3] \rightarrow c[q_2, \#, q_3] \quad [q_2, \#, q_3] \rightarrow \varepsilon$

$[q_1, \#, q_3] \rightarrow a[q_1, A, q_2][q_2, \#, q_3] \quad [q_1, \#, q_3] \rightarrow b[q_1, B, q_2][q_2, \#, q_3]$

$[q_1, A, q_2] \rightarrow a[q_1, A, q_2][q_2, A, q_2] \quad [q_1, A, q_2] \rightarrow b[q_1, B, q_2][q_2, A, q_2]$

$[q_1, B, q_2] \rightarrow a[q_1, A, q_2][q_2, B, q_2] \quad [q_1, A, q_2] \rightarrow b[q_1, B, q_2][q_2, B, q_2]$

$[q_1, A, q_2] \rightarrow c[q_2, A, q_2] \quad [q_1, B, q_2] \rightarrow c[q_2, B, q_2]$

$[q_2, A, q_2] \rightarrow a \quad [q_2, B, q_2] \rightarrow b$

Hopcroft, J. E. and Ullman, J. D. (1979). *Introduction to Automata Theory, Languages and Computation.* Addison Wesley.