

Parsing

Push-Down-Automata (PDA)

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PDA: Intuition (1)

A **push-down automaton** (Hopcroft and Ullman, 1979, 1994) is a **FSA** with an additional stack.

PDA: Intuition (1)

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The moves of the automaton depend on

- the current state
- the next input symbol
- the topmost stack symbol

PDA: Intuition (1)

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The moves of the automaton depend on

- the current state
- the next input symbol
- the topmost stack symbol

Each move consists of

- changing state
- popping the topmost symbol from the stack
- pushing a new sequence of symbols on the stack

PDA: Intuition (2)

Sample PDA

Automaton that

PDA: Intuition (2)

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Automaton that

- starts with q_0 and stack #

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- starts with q_0 and stack $\#$
- in q_0 : pushes A on the stack for an input symbol a
- in q_0 : pushes B on the stack for an input symbol b
- in q_0 : leaves stack unchanged and switches to q_1 for an input symbol c

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- in q_1 : pops an A from the stack for input symbol a
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- in q_1 : moves to q_F if the top of the stack is $\#$

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- accepts all words that allow to end up in q_F

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Sample PDA

Automaton that

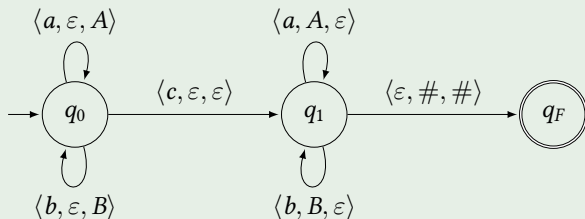
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- accepts all words that allow to end up in q_F

Which (string) language does this automaton accept?

PDA: Intuition (3)

PDAs can also be drawn as graphs:

PDA represented as a graph

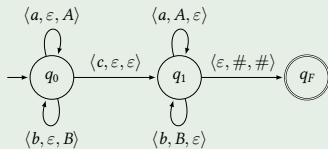


An edge label $\langle a, A, B \rangle$ signifies that a is read, A is popped from the stack and B is pushed on the stack.

PDA: Intuition (4)

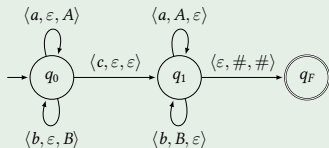
Run of an automaton

Run of this automaton, input aabcbaa:



PDA: Intuition (4)

Run of an automaton

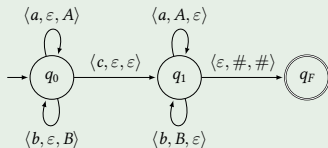


Run of this automaton, input aabcbaa:

state	remaining input	stack	(top on the left)
q_0	aabcbaa	#	

PDA: Intuition (4)

Run of an automaton

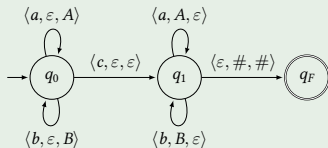


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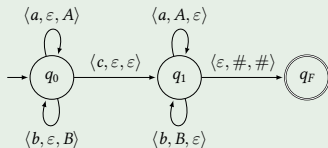


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q_0	bcbaa	AA#	

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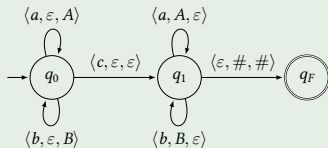


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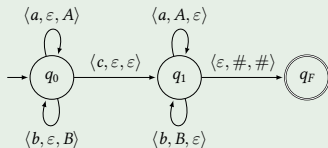


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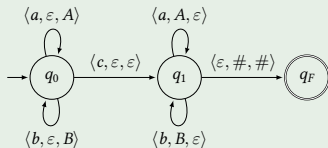


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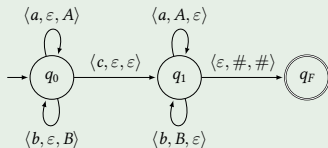


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q_1	aa	AA#	
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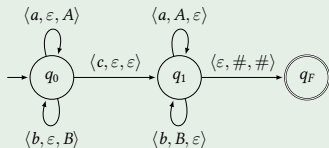


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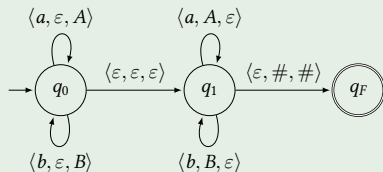
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q_1	aa	AA#	
q_1	a	A#	
q_1	-	#	
q_F	-	#	

PDA: Intuition (5)

In general, PDAs are non-deterministic, since a given state, input symbol and topmost stack symbol can allow for more than one move.

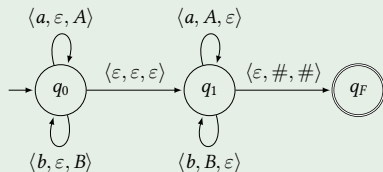
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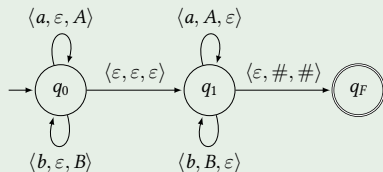


In contrast to FSA, the deterministic version of the automaton is not equivalent to the non-deterministic one: There are languages that are accepted by a non-deterministic PDA but not by any deterministic PDA.

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CFLs are the languages accepted by (non-deterministic) PDAs.

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- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow \mathcal{P}_{fin}(Q \times \Gamma^*)$ is the transition function^a

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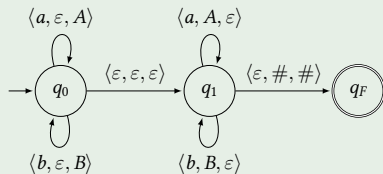
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Equivalently, one can even define δ as

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \rightarrow \mathcal{P}_{fin}(Q \times \Gamma^*).$$

PDA: Definition (3)

Non-deterministic PDA



$Q = \{q_0, q_1, q_F\}$, $\Sigma = \{a, b\}$, $\Gamma = \{\#, A, B\}$, q_0 start state, $F = \{q_F\}$
and

$$\delta(q_0, a, \varepsilon) = \{\langle q_0, A \rangle\} \quad \delta(q_0, b, \varepsilon) = \{\langle q_0, B \rangle\}$$

$$\delta(q_0, \varepsilon, \varepsilon) = \{\langle q_1, \varepsilon \rangle\}$$

$$\delta(q_1, a, B) = \{\langle q_1, \varepsilon \rangle\} \quad \delta(q_1, b, B) = \{\langle q_1, \varepsilon \rangle\}$$

$$\delta(q_1, \varepsilon, \#) = \{\langle q_F, \# \rangle\}$$

In all other cases the value is \emptyset .

PDA: Definition (4)

An **instantaneous description** of a PDA is a triple (q, w, γ) :

- $q \in Q$ is the current state of the automaton
- $w \in \Sigma^*$ is the remaining part of the input string
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For all $q, q' \in Q, a \in \Sigma \cup \{\varepsilon\}, w \in \Sigma^*, \alpha, \beta, \gamma \in \Gamma^*$:

$$(q, aw, \gamma\alpha) \vdash (q', w, \beta\alpha) \text{ iff } \langle q', \beta \rangle \in \delta(q, a, \gamma)$$

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\vdash^* is the reflexive transitive closure of \vdash

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There are two alternatives for the definition of the language accepted by a PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$:

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Language of an PDA

- The **language accepted by M with a final state** is

$$L(M) := \{w \mid (q_0, w, Z_0) \vdash^* (q_f, \varepsilon, \gamma) \text{ for a } q_f \in F \text{ and a } \gamma \in \Gamma^*\}$$

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- The **language accepted by M with an empty stack** is

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The two modes of acceptance are equivalent, i.e., for each language L there is a PDA M_1 with $L = L(M_1)$ iff there is a PDA M_2 with $L = N(M_2)$.

PDA: Definition (6)

Example

PDA $M_1 = \langle Q, \Sigma, \Gamma, \delta, q_0, \#, F \rangle$ for $L(M_1) = \{w c w^R \mid w \in \{a, b\}^*\}$

- $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{\#, A, B\}$, $F = \{q_2\}$.

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■ Transitions:

$$\delta(q_0, a, \varepsilon) = \{\langle q_0, A \rangle\}$$

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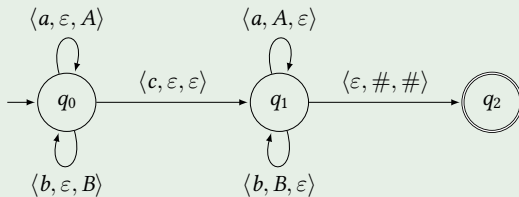
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PDA: Definition (7)

Example

PDA $M_2 = \langle Q, \Sigma, \Gamma, \delta, q_0, \#, F \rangle$ for $N(M_2) = \{wcw^R \mid w \in \{a, b\}^*\}$

PDA: Definition (7)

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- $Q = \{q_0, q_1\}, \Sigma = \{a, b, c\}, \Gamma = \{\#, A, B\}, F = \emptyset.$

PDA: Definition (7)

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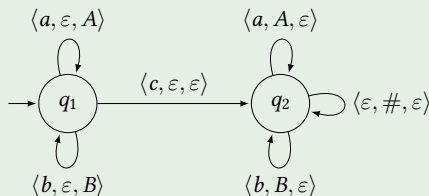
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PDA: Definition (8)

Deterministic PDA

A PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ is a **deterministic PDA** (DPDA) iff

- for all $q \in Q, Z \in \Gamma, a \in \Sigma \cup \{\varepsilon\}$: $|\delta(q, a, Z)| \leq 1$

and

- for all $q \in Q, Z \in \Gamma$: if $\delta(q, \varepsilon, Z) \neq \emptyset$, then $\delta(q, a, Z) = \emptyset$ for all $a \in \Sigma$

(Note that this assumes the definition where

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Examples for DPDA: M_1 and M_2 from the previous slides.

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Deterministic PDA

A PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ is a **deterministic PDA** (DPDA) iff

- for all $q \in Q, Z \in \Gamma, a \in \Sigma \cup \{\varepsilon\}$: $|\delta(q, a, Z)| \leq 1$

and

- for all $q \in Q, Z \in \Gamma$: if $\delta(q, \varepsilon, Z) \neq \emptyset$, then $\delta(q, a, Z) = \emptyset$ for all $a \in \Sigma$

(Note that this assumes the definition where

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow \mathcal{P}_{fin}(Q \times \Gamma^*),$$

i.e., ε as a third argument is not allowed.)

Examples for DPDA: M_1 and M_2 from the previous slides.

The class of languages accepted by DPDAs is smaller than the class accepted by (non-deterministic) PDAs.

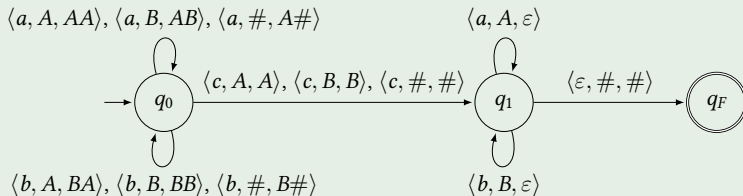
Example of a language that requires a non-deterministic PDA:

$$\{ww^R \mid w \in \{a, b\}^*\}$$

PDA: Definition (9)

Deterministic and non-deterministic PDA

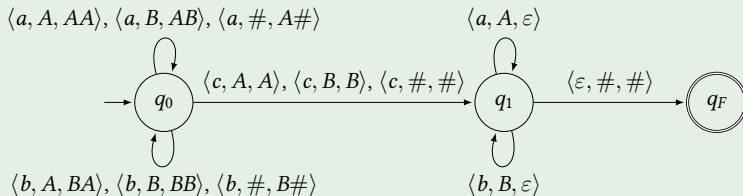
Deterministic PDA for $\{wcw^R \mid w \in \{a, b\}^*\}$:



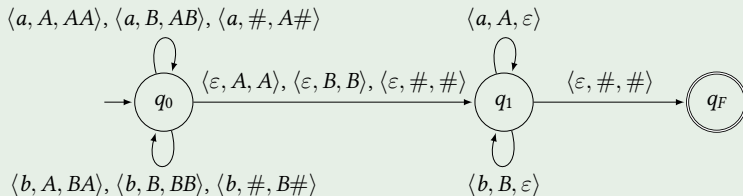
PDA: Definition (9)

Deterministic and non-deterministic PDA

Deterministic PDA for $\{wcw^R \mid w \in \{a, b\}^*\}$:



Non-deterministic PDA for $\{ww^R \mid w \in \{a, b\}^*\}$:



PDA and CFG (1)

For each CFL L , there is a PDA M with $L = N(M)$:

PDA and CFG (1)

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- Assume that $\varepsilon \notin L$

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- $L = L(G)$ for a CFG $G = \langle N, T, P, S \rangle$ in GNF

PDA and CFG (1)

For each CFL L , there is a PDA M with $L = N(M)$:

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- $L = L(G)$ for a CFG $G = \langle N, T, P, S \rangle$ in GNF
- $M = \langle \{q\}, T, N, \delta, q, S, \emptyset \rangle$ with $\langle q, \gamma \rangle \in \delta(q, a, A)$ iff $A \rightarrow a\gamma \in P$

PDA and CFG (1)

For each CFL L , there is a PDA M with $L = N(M)$:

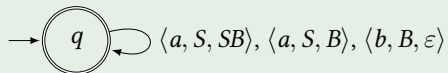
- Assume that $\varepsilon \notin L$
- $L = L(G)$ for a CFG $G = \langle N, T, P, S \rangle$ in GNF
- $M = \langle \{q\}, T, N, \delta, q, S, \emptyset \rangle$ with $\langle q, \gamma \rangle \in \delta(q, a, A)$ iff $A \rightarrow a\gamma \in P$
- The automaton simulates leftmost derivations in G

PDA and CFG (2)

From CFG in GNF to PDA

CFG: $S \rightarrow aSB \mid aB, B \rightarrow b$

Corresponding PDA with S initial stack symbol:

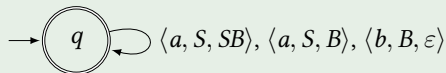


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Run of this automaton:

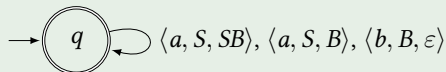
state	remaining input	stack	(top on the left)
q	aabb	S	

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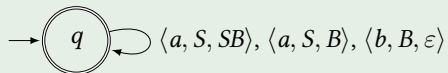
state	remaining input	stack	(top on the left)
q	aabb	S	
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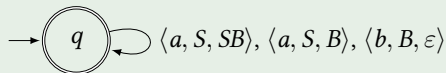
state	remaining input	stack	(top on the left)
q	aabb	S	
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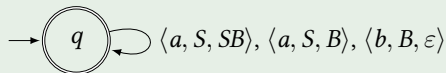
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Run of this automaton:

state	remaining input	stack	(top on the left)
q	aabb	S	
q	abb	SB	
q	bb	BB	
q	b	B	
q	-	-	

PDA and CFG (3)

Two other possibilities to construct a PDA for a CFG $\langle N, T, P, S \rangle$:

Top-down, LL-PDA: We assume that $\langle N, T, P, S \rangle$ is not left-recursive. Then the following PDA accepts the language generated by $\langle N, T, P, S \rangle$:

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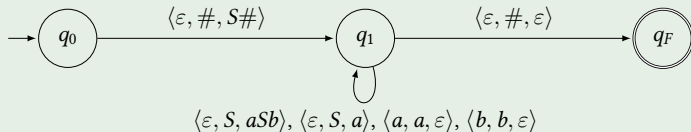
This PDA creates a leftmost derivation when keeping track of the predicted productions in a successful run.

PDA and CFG (4)

LL-PDA

CFG: $S \rightarrow aSb \mid a$

Corresponding PDA with $\#$ initial stack symbol:

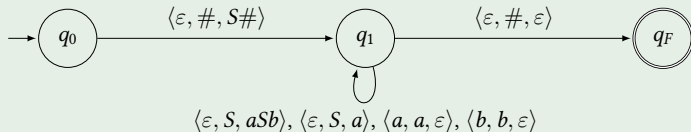


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Run of this automaton:

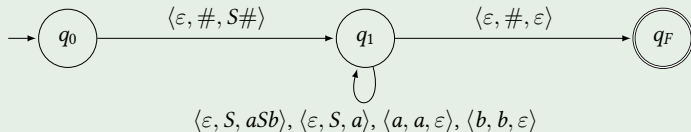
state	remaining input	stack	(top on the left)
q_0	aab	$\#$	

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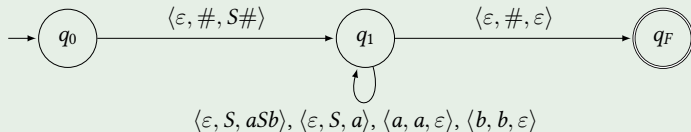
state	remaining input	stack	(top on the left)
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q_1	aab	$S\#$	

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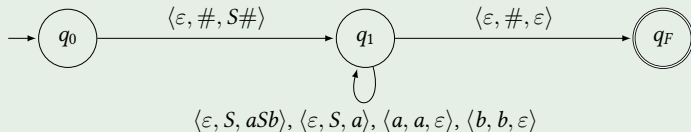
state	remaining input	stack (top on the left)
q_0	aab	$\#$
q_1	aab	$S\#$
q_1	aab	$aSb\#$

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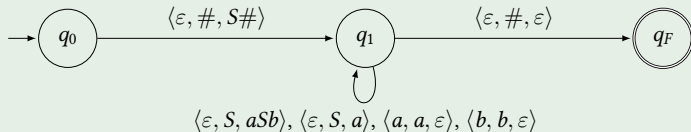
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q_1	ab	$Sb\#$

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Run of this automaton:

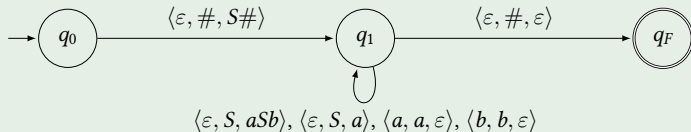
state	remaining input	stack (top on the left)
q_0	aab	$\#$
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q_1	aab	$aSb\#$
q_1	ab	$Sb\#$
q_1	ab	$ab\#$

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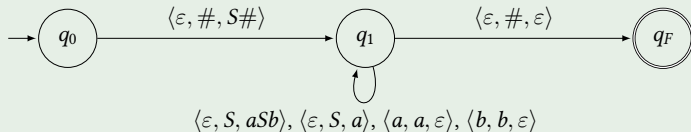
state	remaining input	stack (top on the left)
q_0	aab	$\#$
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q_1	ab	$Sb\#$
q_1	ab	$ab\#$
q_1	b	$b\#$

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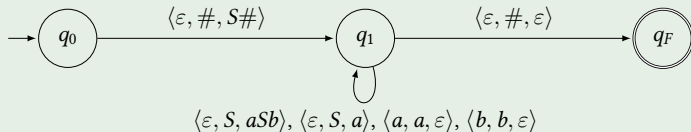
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q_1	ab	$Sb\#$
q_1	ab	$ab\#$
q_1	b	$b\#$
q_1	-	$\#$
q_F	-	-

PDA and CFG (5)

Bottom-up, LR-PDA: We assume that $\langle N, T, P, S \rangle$ does not allow for any loops $A \stackrel{+}{\Rightarrow} A$. Then the following PDA accepts the language generated by $\langle N, T, P, S \rangle$:

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(α^R denotes RHS of production $A \rightarrow \alpha$ in reverse order)

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For this, we use a slightly modified extension of PDA which allows also to pop more than one symbol from the stack in a single transition step. This form of PDA is equivalent to the one we defined above.

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A run of an LR-PDA for a CFG creates a rightmost derivation (in reverse order)

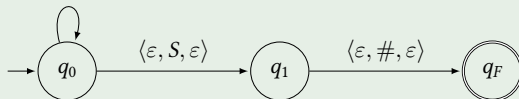
PDA and CFG (6)

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CFG: $S \rightarrow aSb \mid a$

Corresponding PDA with $\#$ initial stack symbol:

$\langle a, \varepsilon, a \rangle, \langle b, \varepsilon, b \rangle, \langle \varepsilon, a, S \rangle, \langle \varepsilon, bSa, S \rangle$



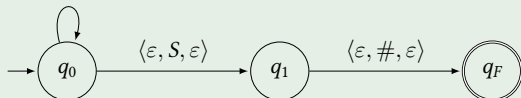
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Corresponding PDA with $\#$ initial stack symbol:

$\langle a, \varepsilon, a \rangle, \langle b, \varepsilon, b \rangle, \langle \varepsilon, a, S \rangle, \langle \varepsilon, bSa, S \rangle$



Run of this automaton:

state	remaining input	stack	(top on the left)
q_0	aab	$\#$	

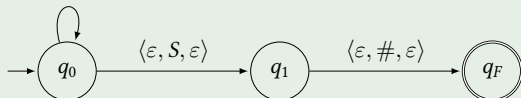
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Run of this automaton:

state	remaining input	stack (top on the left)
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q_0	ab	$a\#$

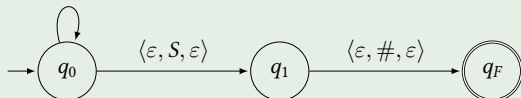
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Run of this automaton:

state	remaining input	stack (top on the left)
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q_0	ab	$a\#$
q_0	b	$aa\#$

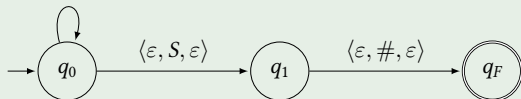
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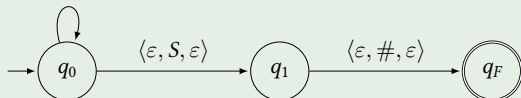
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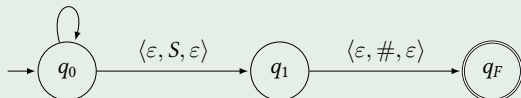
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q_0	ab	$a\#$
q_0	b	$aa\#$
q_0	b	$Sa\#$
q_0	-	$bSa\#$
q_0	-	$S\#$

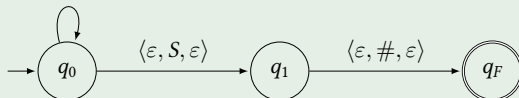
PDA and CFG (6)

LR-PDA

CFG: $S \rightarrow aSb \mid a$

Corresponding PDA with $\#$ initial stack symbol:

$\langle a, \varepsilon, a \rangle, \langle b, \varepsilon, b \rangle, \langle \varepsilon, a, S \rangle, \langle \varepsilon, bSa, S \rangle$



Run of this automaton:

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q_0	aab	$\#$
q_0	ab	$a\#$
q_0	b	$aa\#$
q_0	b	$Sa\#$
q_0	-	$bSa\#$
q_0	-	$S\#$
q_1	-	$\#$

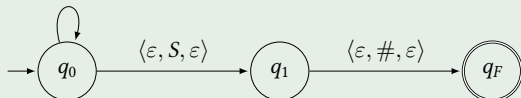
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q_F	-	-

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It holds: $[q_1, A, q_2] \xRightarrow{*} w$ iff $(q_1, w, A) \vdash^* (q_2, \varepsilon, \varepsilon)$

- Hopcroft, J. E. and Ullman, J. D. (1979). *Introduction to Automata Theory, Languages and Computation*. Addison Wesley.
- Hopcroft, J. E. and Ullman, J. D. (1994). *Einführung in die Automatentheorie, Formale Sprachen und Komplexitätstheorie*. Addison Wesley, 3. edition.