Parsing

Context-Free Grammars (CFG)

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Type 1/2/3 grammars

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The hierarchy of the type 0, 1, 2 and 3 languages is called the **Chomsky Hierarchy**.

Grammar Formalisms (again)

Type 3 grammar

Grammar for L(das (rote|grüne)* Auto (von Otto| ε))

NP $\rightarrow$ Det N1	N1 $\rightarrow$ rote N1	N1 $\rightarrow$ grüne N1	N1 $\rightarrow$ Auto
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Type 2 grammar

Grammar for $\{a^n b^m (cd)^n d \mid n, m \geq 0\}$

S $\rightarrow$ T d T $\rightarrow$ a T cd T $\rightarrow$ U U $\rightarrow$ b U U $\rightarrow$ ε

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$S \rightarrow T\ d$ $T \rightarrow a\ T\ cd$ $T \rightarrow U$ $U \rightarrow b\ U$ $U \rightarrow \varepsilon$

Type 1 grammar

Grammar for $\{ww \mid w \in \{a, b\}^+\}$

$S \rightarrow a\ S\ A$ $S \rightarrow b\ S\ B$ $S \rightarrow A_f\ A$ $S \rightarrow B_f\ B$
$a\ A \rightarrow A\ a$ $b\ A \rightarrow A\ b$ $a\ B \rightarrow B\ a$ $b\ B \rightarrow B\ b$
$A_f\ A \rightarrow A_f\ a$ $B_f\ B \rightarrow B_f\ b$ $A_f\ B \rightarrow A_f\ b$ $B_f\ A \rightarrow B_f\ a$
$A_f \rightarrow a$ $B_f \rightarrow b$

CFG

CFG

A **context-free grammar** (CFG) is a tuple $G = \langle N, T, P, S \rangle$ such that:

- N and T are disjoint alphabets, the nonterminals and terminals
- $S \in N$ is the start symbol
- P is a set of productions of the form $A \to \beta$ with $A \in N, \beta \in (N \cup T)^*$

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Any $\beta \in (N \cup T)^*$ with $S \stackrel{*}{\Rightarrow} \beta$ is called a **sentential form**.

CFG

CFG parse tree

A tree t is a **parse tree** for a CFG $G = \langle N, T, P, S \rangle$ iff

- each node in t is labeled with an $\alpha \in N \cup T \cup \{\varepsilon\}$
- the root label is S
- if there is a node with label A that has n daughters labeled (from left to right) $\alpha_1, \ldots, \alpha_n$, then $A \rightarrow \alpha_1 \ldots \alpha_n \in P$
- if a node has label ε, it is a leaf and the unique daughter of its mother node

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$S \stackrel{*}{\Rightarrow} \beta$ in G iff there is a parse tree for G with yield β.

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Languages generated by a CFG

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  1. for $w, w' \in (N \cup T)^*$: $w \Rightarrow w'$ iff there is a $A \to \beta \in P$ and there are $v, u \in (N \cup T)^*$ such that $w = vAu$ and $w' = v\beta u$.

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  2. $\stackrel{*}{\Rightarrow}$ is the reflexive transitive closure of $\Rightarrow$.
- A **derivation** of a word $w \in T^*$ is a sequence

$$S \Rightarrow \alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow w$$

of derivation steps leading to w.

CFG

For a single parse tree, there might be more than one corresponding derivation.

Derivations

CFG $G_{a,b} = \langle \{S, A, B\}, \{a, b\}, P, S \rangle$ with productions

$$S \to AB \mid BA \qquad A \to a \mid aS \mid bAA \qquad B \to b \mid bS \mid aBB$$

(This CFG generates the language $\{w \in \{a, b\}^+ \mid |w|_a = |w|_b\}$.)

Input $w = ab$.

The two derivations for w are
$S \Rightarrow AB \Rightarrow aB \Rightarrow ab$ and $S \Rightarrow AB \Rightarrow Ab \Rightarrow ab$

($|w|_a$ gives the number of occurrences of a in w.)

CFG

Leftmost and rightmost derivations

A derivation is called a

- **leftmost** derivation iff, in each derivation step, a production is applied to the leftmost non-terminal of the already derived sentential form
- **rightmost** derivation iff, in each derivation step, a production is applied to the rightmost non-terminal of the already derived sentential form

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In the preceding example, the first derivation was a leftmost derivation and the second a rightmost derivation.

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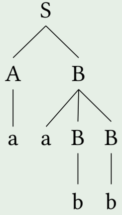
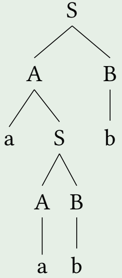
Ambiguous grammar

Consider again $G_{a,b}$ ($S \to AB \mid BA$, $A \to a \mid aS \mid bAA$, $B \to b \mid bS \mid aBB$)

The two parse trees for *aabb* are





CFG

Some languages are such that their structure is necessarily ambiguous. Natural languages are probably such cases.

Inherently ambiguous

A CFL L is called **inherently ambiguous** if each CFG G with $L = L(G)$ is ambiguous.

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Inherently ambiguous language

$$L = \{a^n b^n c^m d^m \mid n \geq 1, m \geq 1\} \cup \{a^n b^m c^m d^n \mid n \geq 1, m \geq 1\}$$

For words of the form $a^k b^k c^k d^k$ one cannot tell wich of the two patterns is the right structure. Both are possible.

Removing useless symbols

An important grammar clean-up one has to do sometimes is the removal of symbols (non-terminals or terminals) that cannot occur in any derivation of a word in the string language.

Useful/useless symbols

Let $G = \langle N, T, P, S \rangle$ be a CFG. An $X \in N \cup T$ is called

- **useful** if there is a derivation $S \overset{*}{\Rightarrow} \alpha X \beta \overset{*}{\Rightarrow} w$ with $w \in T^*$
- **useless** otherwise

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- Non-terminals X can be useless because they do not allow to derive a terminal string, i.e., there is no derivation $X \stackrel{*}{\Rightarrow} w$ with $w \in T^*$.
- Non-terminals and terminals X can be useless because they cannot be reached from the start symbol, i.e., there is no derivation $S \stackrel{*}{\Rightarrow} \alpha X$.

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Useless symbols

1. $G = \langle\{S, A, B, C\}, \{a, b, c\}, P, S\rangle$ with
   $P = \{S \rightarrow AB \,|\, aS, A \rightarrow aaAc \,|\, c, B \rightarrow aCc\}$
   Useless: B and C since from them one cannot derive any terminal string.
2. $G = \langle\{S, A, B, C\}, \{a, b, c\}, P, S\rangle$ with
   $P = \{S \rightarrow AB \,|\, aS, A \rightarrow aaAc \,|\, c, B \rightarrow ac, C \rightarrow b\}$
   Useless: C and b since they are not reachable from S.

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1. All $X \in N$ need to be eliminated that cannot lead to a terminal sequence.
   This can be done recursively: Starting from the terminals and following the productions from right to left, the set of all symbols leading to terminals can be computed recursively.
   Productions containing symbols that are not in this set are eliminated.

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2. In the resulting CFG, the unreachable symbols need to be eliminated.
   This is done starting from S and applying productions. Each time, the symbols from the right-hand sides are added.
   Again, productions containing non-terminals or terminals that are not in the set are eliminated.

Removing useless symbols

Removing useless symbols

$G = \langle \{S, A, B, C, D, E\}, \{a, b, c\}, P, S \rangle$ with
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Resultig CFG: $\langle \{A, B, S\}, \{a, b\}, \{S \rightarrow AB, A \rightarrow a, B \rightarrow bb\}, S\rangle$

Eliminating ε-rules

Let $G = \langle N, T, P, S \rangle$. A production of the form $A \rightarrow \varepsilon$ is called an ε-production.

The following holds:

For each CFG G, there is a CFG G' without ε-productions such that $L(G') = L(G) \setminus \{\varepsilon\}$.

Removing ε-productions

$G = \langle \{S, T\}, \{a, b, c, d\}, P, S \rangle$ with
$P = \{S \rightarrow aSb \mid aTb, T \rightarrow cTd \mid \varepsilon\}$

Equivalent ε-free CFG:
$G = \langle \{S, T\}, \{a, b, c, d\}, P, S \rangle$ with
$P = \{S \rightarrow aSb \mid aTb \mid ab, T \rightarrow cTd \mid cd\}$

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  1. $N_\varepsilon := \{A \in N \mid A \Rightarrow \varepsilon\}$
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    3. Repeat 2. until N_ε does not change any more
- Delete the ε-productions and for each $A \rightarrow X_1 \dots X_n$:
  add all productions one can obtain by deleting some $X_j \in N_\varepsilon$ from the right-hand side as long as one does not delete all $X_1, \dots, X_n$.

Removing unary rules

Let $G = \langle N, T, P, S \rangle$. For $A, B \in N$, a production of the form $A \rightarrow B$ is called a unary production

For each CFL that does not contain ε-rules, a CFG without unary productions can be found.

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Elimination of unary productions for a CFG without ε-productions:

1. For all $A \stackrel{*}{\Rightarrow} B$ and all $B \rightarrow \beta, \beta \notin N$:
   add $A \rightarrow \beta$ to the set of productions
2. Delete all unary productions

Chomsky Normal Form

A normal form of a grammar formalism F is a further restriction on the grammars in F that does not affect the set of generated string languages.

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There are two important normal forms for CFGs.

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2. in **Greibach normal form** iff all productions have the form $A \rightarrow a\alpha$ with $a \in T$, $\alpha \in N^*$

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For each CFL L without ε, there is a CFG in Chomsky normal form with $L = L(G)$.

Construction of an equivalent CFG in CNF for a given CFG $G = \langle N, T, P, S \rangle$

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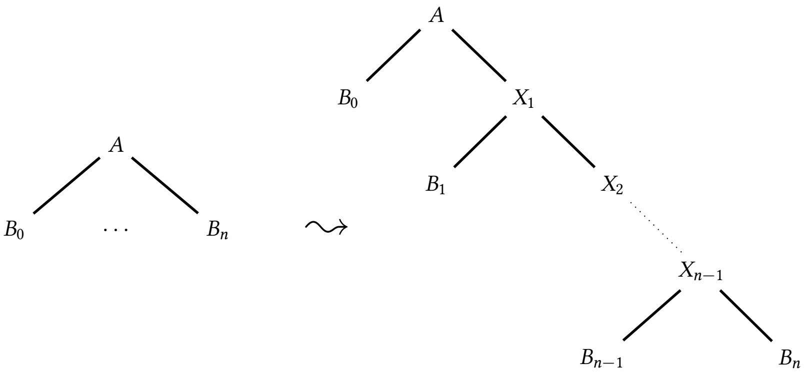
5. For each production $A \to B_0 \dots B_n$ introduce new non-terminals $X_1, \dots, X_{n-1}$ and replace production $A \to B_0 \dots B_n$ with

$$A \to B_0 X_1 \qquad X_1 \to B_1 X_2 \qquad \dots \qquad X_{n-1} \to B_{n-1} B_n$$

Chomsky Normal Form



Chomsky Normal Form

Transformation to CNF

$G = \langle\{\mathrm{S,C}\}, \{\mathrm{a,b,c,}\}, \{\mathrm{S} \rightarrow \mathrm{aSbC} \mid \mathrm{ab},\ \mathrm{C} \rightarrow \mathrm{c} \mid \mathrm{cC}\}, \mathrm{S}\rangle.$

Chomsky Normal Form

Transformation to CNF

$G = \langle \{S,C\}, \{a,b,c,\}, \{S \to aSbC \mid ab, C \to c \mid cC\}, S\rangle.$

1. Introducing new preterminals:
   $G = \langle \{S, C, T_a, T_b, T_c\}, \{a, b, c\}, P, S\rangle$ with productions
   $S \to \mathbf{T_a} S \mathbf{T_b} C \mid \mathbf{T_a}\mathbf{T_b}, C \to c \mid \mathbf{T_c} C$

Chomsky Normal Form

Transformation to CNF

$G = \langle \{S,C\}, \{a,b,c,\}, \{S \to aSbC \mid ab,\ C \to c \mid cC\}, S \rangle$.

1. Introducing new preterminals:
   $G = \langle \{S, C, T_a, T_b, T_c\}, \{a, b, c\}, P, S \rangle$ with productions
   $S \to \mathbf{T_a}S\mathbf{T_b}C \mid \mathbf{T_a}\mathbf{T_b},\ C \to c \mid \mathbf{T_c}C,\ \mathbf{T_a} \to a,\ \mathbf{T_b} \to b,\ \mathbf{T_c} \to c$

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   $S \to \mathbf{T_a}S\mathbf{T_b}C \mid \mathbf{T_a}\mathbf{T_b}, C \to c \mid \mathbf{T_c}C, \mathbf{T_a} \to a, \mathbf{T_b} \to b, \mathbf{T_c} \to c$

2. Binarization:
   $G = \langle \{S, C, T_a, T_b, T_c, X_1, X_2\}, \{a, b, c\}, P, S \rangle$ with productions
   $S \to \mathbf{T_a X_1} \mid T_a T_b, \mathbf{X_1} \to \mathbf{S X_2}, \mathbf{X_2} \to \mathbf{T_b C}, C \to c \mid T_c C,$
   $T_a \to a, T_b \to b, T_c \to c$

Greibach Normal Form

For each CFL L without ε, there is a CFG in Greibach normal form with $L = L(G)$.

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2. Eliminate unary productions
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Left-recursive CFG

A CFG $G = \langle N, T, P, S \rangle$ is **left-recursive** iff there is a non-terminal $A \in N$ and a sequence $\alpha \in (N \cup T)^+$ such that $A \stackrel{+}{\Rightarrow} A\alpha$

Greibach Normal Form

Elimination of left-recursion

- Assume the set of non-terminals to be ordered, i.e. $N = \{A_1, \ldots, A_m\}$

Greibach Normal Form

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- Assume the set of non-terminals to be ordered, i.e. $N = \{A_1, \ldots, A_m\}$
- Construct a CFG with $j > i$ if $A_i \rightarrow A_j\gamma$:

  For all A_i, $1 \leq i \leq m$, steps (I) and (II) are done.

(I) Transformation such that $j \geq i$ if $A_i \rightarrow A_j\gamma$:

Replace all productions $A_i \rightarrow A_j\gamma$ with $j < i$ with new productions obtained from replacing A_j with all right-hand sides of A_j-productions. Do this until the condition holds for all A_i-productions.

Greibach Normal Form

Elimination of left-recursion I

$S \to AB \quad A \to S \quad A \to a \quad B \to b$

Greibach Normal Form

Elimination of left-recursion I

$S \rightarrow AB \quad A \rightarrow S \quad A \rightarrow a \quad B \rightarrow b$

We put indices on our non-terminals: B has index 1, A index 2 and S index 3 (other indexations work as well but lead to a different grammar)

$S_3 \rightarrow A_2 B_1 \quad A_2 \rightarrow S_3 \quad A_2 \rightarrow a \quad B_1 \rightarrow b$

Greibach Normal Form

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Problematic production with a righthand side (rhs) starting with an index lower than the lefthand side index: $S_3 \to A_2B_1$

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Problematic production with a righthand side (rhs) starting with an index lower than the lefthand side index: $S_3 \to A_2B_1$

Replace A_2 in this rule with the rhs of A_2-productions. Our new productions are

$S_3 \to S_3B_1 \quad S_3 \to aB_1 \quad A_2 \to S_3 \quad A_2 \to a \quad B_1 \to b$

Greibach Normal Form

Idea behind the next step:

- Assume that for A, we have productions

$$A \to A\alpha_1,\ \ldots,\ A \to A\alpha_r,\ A \to \beta_1,\ \ldots,\ A \to \beta_s$$

- This means that we can have a derivation

$$A \Rightarrow A\alpha_{i_1} \Rightarrow A\alpha_{i_1}\alpha_{i_2} \Rightarrow \cdots A\alpha_{i_1} \ldots \alpha_{i_n} \Rightarrow \beta_j\alpha_{i_1} \ldots \alpha_{i_n}$$

  where the α-parts are from $\{\alpha_1, \ldots, \alpha_r\}$ and the β-part is from $\{\beta_1, \ldots, \beta_s\}$.

- We can also generate this by a right-recursion using a new non-terminal B that generates a sequence of elements from $\{\alpha_1, \ldots, \alpha_r\}$:

$$A \Rightarrow \beta_j B \Rightarrow \beta_j\alpha_{i_1} B \Rightarrow \beta_j\alpha_{i_1}\alpha_{i_2} B \Rightarrow \cdots \Rightarrow \beta_j\alpha_{i_1} \ldots \alpha_{i_n}$$

Greibach Normal Form

(II) Elimination of left-recursive productions $A_i \to A_i\alpha$:

Add a new non-terminal B and replace

$$A_i \to A_i\alpha_1,\ \ldots,\ A_i \to A_i\alpha_r,\ A_i \to \beta_1,\ \ldots,\ A_i \to \beta_s$$

with

$$A_i \to \beta_i,\ A_i \to \beta_i B \quad \forall i,\ 1 \leq i \leq s$$

and

$$B \to \alpha_i,\ B \to \alpha_i B \quad \forall i,\ 1 \leq i \leq r$$

Greibach Normal Form

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with

$$A_i \to \beta_i,\ A_i \to \beta_i B \quad \forall i,\ 1 \leq i \leq s$$

and

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(Left recursion is turned into right recursion)

Greibach Normal Form

(II) Elimination of left-recursive productions $A_i \to A_i\alpha$:

Add a new non-terminal B and replace

$$A_i \to A_i\alpha_1, \ \ldots, \ A_i \to A_i\alpha_r, \ A_i \to \beta_1, \ \ldots, \ A_i \to \beta_s$$

with

$$A_i \to \beta_i, \ A_i \to \beta_i B \quad \forall i, \ 1 \leq i \leq s$$

and

$$B \to \alpha_i, \ B \to \alpha_i B \quad \forall i, \ 1 \leq i \leq r$$

(Left recursion is turned into right recursion)

In the resulting grammar, for all $A_i \stackrel{+}{\Rightarrow} A_j\alpha$, $i < j$ holds.

Greibach Normal Form

Elimination of left-recursion II

$S_3 \to S_3 B_1 \quad S_3 \to aB_1 \quad A_2 \to S_3 \quad A_2 \to a \quad B_1 \to b$

- We introduce a new non-terminal C and
- replace $S_3 \to S_3 B_1$, $S_3 \to aB_1$
  with $S_3 \to aB_1$, $S_3 \to aB_1 C$, $C \to B_1 C$, $C \to B_1$.

Result:

$S_3 \to aB_1 \quad S_3 \to aB_1 C \quad C \to B_1 C \quad C \to B_1$

$A_2 \to S_3 \quad A_2 \to a \quad B_1 \to b$

Greibach Normal Form

Elimination of left-recursion II

$S_3 \to S_3B_1 \quad S_3 \to aB_1 \quad A_2 \to S_3 \quad A_2 \to a \quad B_1 \to b$

- We introduce a new non-terminal C and
- replace $S_3 \to S_3B_1$, $S_3 \to aB_1$
  with $S_3 \to aB_1$, $S_3 \to aB_1C$, $C \to B_1C$, $C \to B_1$.

Result:

$S_3 \to aB_1 \quad S_3 \to aB_1C \quad C \to B_1C \quad C \to B_1$

$A_2 \to S_3 \quad A_2 \to a \quad B_1 \to b$

After further removal of the now useless A_2 and of the unary production, we obtain

$S_3 \to aB_1 \quad S_3 \to aB_1C \quad C \to B_1C \quad C \to b \quad B_1 \to b$

Greibach Normal Form

Elimination of left-recursion

We could also start with different indices, e.g.,

$$S_2 \to A_3 B_1 \quad A_3 \to S_2 \quad A_3 \to a \quad B_1 \to b$$

Greibach Normal Form

Elimination of left-recursion

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$$S_2 \to A_3 B_1 \quad A_3 \to S_2 \quad A_3 \to a \quad B_1 \to b$$

After step I, we would have

$$S_2 \to A_3 B_1 \quad A_3 \to A_3 B_1 \quad A_3 \to a \quad B_1 \to b$$

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Elimination of left-recursion

We could also start with different indices, e.g.,

$$S_2 \rightarrow A_3B_1 \quad A_3 \rightarrow S_2 \quad A_3 \rightarrow a \quad B_1 \rightarrow b$$

After step I, we would have

$$S_2 \rightarrow A_3B_1 \quad A_3 \rightarrow A_3B_1 \quad A_3 \rightarrow a \quad B_1 \rightarrow b$$

After step II, the result would be

$$S_2 \rightarrow A_3B_1 \quad A_3 \rightarrow a \quad A_3 \rightarrow aC \quad C \rightarrow B_1 \quad C \rightarrow B_1C \quad B_1 \rightarrow b$$

Greibach Normal Form

Elimination of left-recursion

We could also start with different indices, e.g.,

$$S_2 \to A_3 B_1 \quad A_3 \to S_2 \quad A_3 \to a \quad B_1 \to b$$

After step I, we would have

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After step II, the result would be

$$S_2 \to A_3 B_1 \quad A_3 \to a \quad A_3 \to aC \quad C \to B_1 \quad C \to B_1 C \quad B_1 \to b$$

After elimination of the unary production $C \to B_1$, this yields

$$S_2 \to A_3 B_1 \quad A_3 \to a \quad A_3 \to aC \quad C \to b \quad C \to B_1 C \quad B_1 \to b$$

Greibach Normal Form

Transformation to GNF

$G = \langle N, T, P, S \rangle,\ N = \{A_1, A_2, A_3, B\},\ T = \{a, b\}$

$A_1 \rightarrow A_2A_3 \quad A_2 \rightarrow A_3A_1 \mid b \quad A_3 \rightarrow A_1A_2 \mid a$

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$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_1A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_2A_3A_2 \mid a$$

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$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_2A_3A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_3A_1A_3A_2 \mid bA_3A_2 \mid a$$

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$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_1A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_2A_3A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_3A_1A_3A_2 \mid bA_3A_2 \mid a$$

Replace $A_3 \to A_3A_1A_3A_2 \mid bA_3A_2 \mid a$ by

$$A_3 \to bA_3A_2 \qquad A_3 \to a \qquad A_3 \to bA_3A_2B \quad A_3 \to aB$$
$$B \to A_1A_3A_2 \quad B \to A_1A_3A_2B$$

Greibach Normal Form

Transformation to GNF

$G = \langle N, T, P, S \rangle,\ N = \{A_1, A_2, A_3, B\},\ T = \{a, b\}$

$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_1A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_2A_3A_2 \mid a$$
$$A_1 \to A_2A_3 \quad A_2 \to A_3A_1 \mid b \quad A_3 \to A_3A_1A_3A_2 \mid bA_3A_2 \mid a$$

Replace $A_3 \to A_3A_1A_3A_2 \mid bA_3A_2 \mid a$ by

$$A_3 \to bA_3A_2 \qquad A_3 \to a \qquad A_3 \to bA_3A_2B \quad A_3 \to aB$$
$$B \to A_1A_3A_2 \quad B \to A_1A_3A_2B$$

The resulting grammar does not allow left-recursive derivations.

Greibach Normal Form

Lexicalization

- Now, two more steps are necessary to obtain the grammar in GNF:

Greibach Normal Form

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  5. For $i = m - 1$ to $i = 1$: Replace all $A_i \to A_j\beta$ with all productions obtained by replacing A_j with the right-hand side of a A_j-production

     Then, do the same for all B-productions where B is one of the symbols introduced in step (II)

Greibach Normal Form

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- Now, two more steps are necessary to obtain the grammar in GNF:
  5. For $i = m - 1$ to $i = 1$: Replace all $A_i \rightarrow A_j\beta$ with all productions obtained by replacing A_j with the right-hand side of a A_j-production

     Then, do the same for all B-productions where B is one of the symbols introduced in step (II)
  6. For all productions $A \rightarrow \alpha a\beta, \alpha \neq \varepsilon$: replace a with a new non-terminal T_a and add a production $T_a \rightarrow a$ to P.

Greibach Normal Form

Transformation to GNF (cont'd)

$A_1 \to A_2 A_3$

$A_2 \to A_3 A_1 \mid b$

$A_3 \to bA_3A_2 \mid a \mid bA_3A_2B \mid aB$

$B \to A_1A_3A_2 \mid A_1A_3A_2B$

Greibach Normal Form

Transformation to GNF (cont'd)

$A_1 \to A_2 A_3$
$A_2 \to A_3 A_1 \mid b$
$A_3 \to b A_3 A_2 \mid a \mid b A_3 A_2 B \mid aB$
$B \to A_1 A_3 A_2 \mid A_1 A_3 A_2 B$

Replace

1. $A_2 \to A_3 A_1$ by
   $A_2 \to b A_3 A_2 A_1 \mid a A_1 \mid b A_3 A_2 B A_1 \mid a B A_1$

Greibach Normal Form

Transformation to GNF (cont'd)

$A_1 \to A_2 A_3$
$A_2 \to A_3 A_1 \mid b$
$A_3 \to b A_3 A_2 \mid a \mid b A_3 A_2 B \mid aB$
$B \to A_1 A_3 A_2 \mid A_1 A_3 A_2 B$

Replace

1. $A_2 \to A_3 A_1$ by
   $A_2 \to b A_3 A_2 A_1 \mid a A_1 \mid b A_3 A_2 B A_1 \mid a B A_1$
2. $A_1 \to A_2 A_3$ by
   $A_1 \to b A_3 A_2 A_1 A_3 \mid \ldots \mid a B A_1 A_3 \mid b A_3$

Greibach Normal Form

Transformation to GNF (cont'd)

$$A_1 \to A_2A_3$$
$$A_2 \to A_3A_1 \mid b$$
$$A_3 \to bA_3A_2 \mid a \mid bA_3A_2B \mid aB$$
$$B \to A_1A_3A_2 \mid A_1A_3A_2B$$

Replace

1. $A_2 \to A_3A_1$ by
   $A_2 \to bA_3A_2A_1 \mid aA_1 \mid bA_3A_2BA_1 \mid aBA_1$
2. $A_1 \to A_2A_3$ by
   $A_1 \to bA_3A_2A_1A_3 \mid \ldots \mid aBA_1A_3 \mid bA_3$
3. $B \to A_1A_3A_2$ by
   $B \to bA_3A_2A_1A_3A_3A_2 \mid \ldots \mid bA_3A_3A_2$

Greibach Normal Form

Transformation to GNF (cont'd)

$A_1 \to A_2A_3$
$A_2 \to A_3A_1 \mid b$
$A_3 \to bA_3A_2 \mid a \mid bA_3A_2B \mid aB$
$B \to A_1A_3A_2 \mid A_1A_3A_2B$

Replace

1. $A_2 \to A_3A_1$ by
   $A_2 \to bA_3A_2A_1 \mid aA_1 \mid bA_3A_2BA_1 \mid aBA_1$
2. $A_1 \to A_2A_3$ by
   $A_1 \to bA_3A_2A_1A_3 \mid \ldots \mid aBA_1A_3 \mid bA_3$
3. $B \to A_1A_3A_2$ by
   $B \to bA_3A_2A_1A_3A_3A_2 \mid \ldots \mid bA_3A_3A_2$
4. $B \to A_1A_3A_2B$ by
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