Benedikt **Löwe**, Volker **Peckhaus**, Thoralf **Räsch** (*eds.*)
Foundations of the Formal Sciences IV
The History of the Concept of the Formal Sciences
Papers of the conference held in Bonn, February 14-17, 2003

On Formal Objects

Rainer Osswald

Praktische Informatik VII
Informatikzentrum
FernUniversität in Hagen
Universitätsstraße 1
58084 Hagen
Germany

E-mail: `rainer.osswald@fernuni-hagen.de`

Abstract. One of the most elaborate approaches to specifying formal systems and objects has been given by H. B. Curry. Although Curry is mainly known as a proponent of the formalist viewpoint on mathematics, his conception of a formal system turns out to show constructivist and structuralist aspects as well. In particular, Curry emphasizes that the exact nature of mathematical objects is irrelevant with respect to the truth of mathematical statements. This view is in accordance with a Quinean conception of structuralism, which comes along with a relative notion of ontology.

On the other hand, there is a crucial difference in attitude between Curry and Quine concerning their ontological commitments in specifying formal objects; for example, Curry regards inductively generated classes of objects as intuitively given by means of inductive specifications, whereas Quine abstains from drawing on the intuitively given when it comes to ontology. This conflict is closely related to differing conceptions of logic. For Quine, in contrast to Curry, elementary logic is an indispensable part of any serious science – including the science of formal systems.

Received: ...;
In revised version: ...;
Accepted by the editors: ....
2000 *Mathematics Subject Classification.* **PRIMARY** SECONDARY.

1 Introduction

By a formal object we mean things like numbers, strings, tuples, lists, trees, or the abstract data types used in computer science. The starting point of our discussion will be Haskell B. Curry's view of formal objects. Curry is generally considered to be the most prominent (if not the only) proponent of a formalist philosophy of mathematics in the second half of the twentieth century. Without doubt, Curry's elaboration of the formalist position is the most detailed one up to date. The standard reference is his 1951 booklet *Outlines of a Formalist Philosophy of Mathematics* [Cur51], which was already written in 1939.[1] In subsequent years, Curry further developed and modified his approach, culminated in his *Foundations of Mathematical Logic* [Cur63], which we take as our primary reference.

Curry's formalist philosophy has its origins in the ideas of the Hilbert school, which he got acquainted with during a stay in Göttingen.[2] W. V. Quine, on the other hand, is often associated with "logicism" in that he stands in the tradition of Frege, Russell, and Carnap. Although such a classification may be more irritable than useful, it correctly hints at the central role of logic in Quine's philosophy. He regards logic as the grammar of science, which includes of course the formal sciences as well. In what follows, we will try to spell out the different positions of Curry and Quine with respect to logic and ontology and their consequences concerning the nature of formal objects.

2 Formal Systems and Objects

This section briefly introduces the basic notions of Curry's conception of a formal system. A discussion of the underlying assumptions will follow below when Curry's approach is analyzed from a Quinean viewpoint. If not otherwise indicated, page numbers refer to [Cur63] in this section.

---

[1] as an invited paper for an International Congress for the Unity of Science in the same year; see [SH80, p. v].

[2] He wrote his dissertation *Grundlagen der kombinatorischen Logik* (1929) under Hilbert, although he did most of his work with Paul Bernays; cf. [SH80, p. viii].

2.1 Classes and Processes

Since Curry aims at foundations, he abstains from presupposing anything like axiomatic set theory, which he regards as a higher part of logic. Nevertheless, in specifying formal systems he needs to refer to certain "totalities":

> We shall often have to formulate [...] properties (or relations) which define, in a strictly intuitive (or contensive) way, a totality of elements of notions. In order to distinguish such intuitive totalities from the "sets" or "classes" formed later (and conceived rather as objects of some theoretical study than as intuitive notions), we shall call them *conceptual classes* (or relations). [p. 38]

A conceptual class is thus a totality of elements defined in a strictly intuitive or contensive way.[3] Such a class is said to be *definite* if the question of membership can be decided by an effective process.

An *effective process* for attaining a certain goal for a given element is a sequence of transformations to be applied successively to the element such that the goal is reached after a finite number of steps (cf. p. 37). In particular, one needs to know which elements are *admissible* for which transformations and what the results of the latter will be. Contrasting his notion of an effective process with other constructivist programs, like that of the inuitionists, Curry states that his notion "does not depend on any idealistic intuition, temporal of otherwise." Although a critical analysis is postponed to later sections, we can at least note that if not on idealistic intuitions, his proposal heavily relies on non-idealistic ones.

A conceptual class is called *inductive* if it "is generated from certain initial elements by certain specified modes of combination" [p. 38]. More precisely, an inductive class X is given by *initial* and *generating specifications*. The initial specifications determine a definite class of *initial objects*, the *basis* of X; the generating specifications define a definite class of *modes of combination* of finite degree, each of which when applied to a tuple of elements of X produces an element of X. In addition, an inductive class is assumed to satisfy the *closure specification*, which says that "every element [of the class] can be reached by an effective process [...] which starts with certain initial elements and at each later step applies a mode of combination to arguments already constructed" [p. 39].

A *construction* of an element of an inductive class is a process for reaching that element by iterated application of the modes of combination. An inductive class is called *monotectonic*, if every element has a

[3] 'Contensive' is Curry's translation of 'inhaltlich'.

unique construction, and *polytectonic*, otherwise. The prototypical example for the latter type is given by the inductive class of finite strings over some alphabet, with concatenation of strings as the single mode of combination.

2.2 Formal Systems

In short, a formal system is a theory about formal objects.[4] More exactly, a *formal system* is given by a conceptual class of *formal objects*, a conceptual class of *basic predicates* of finite degree, and a conceptual class of *elementary statements*, the *elementary theorems*, which assert that certain basic predicates hold of certain formal objects [p. 50]. The system is called *deductive*, if the class of theorems is inductive, in which case the initial elements are referred to as *axioms* and the modes of combination as *deductive rules* [p. 46]. In case the class of basic predicates is empty, Curry speaks of the formal system as *pure morphology*.

Curry distinguishes between *syntactical systems* and *ob systems* (pp. 51ff). In syntactical systems, the formal objects are the *finite strings* over the *letters* of some *alphabet*. There are two principal ways of conceiving these formal objects as an inductive class. The first option is to use, for each letter, the *affixation* of that letter to the right as a (unary) mode of combination, and the letters as initial elements. The second option employs concatenation as a single (binary) mode of combination, where again the letters serve as initial elements. Notice that affixation leads to a monotectonic inductive class whereas concatenation gives rise to a polytectonic one.

An *ob system* is based on a monotectonic inductive class of formal objects, called *obs*; the initial elements are called *atoms*, the modes of combination *operations*. Observe that an affixative syntactical system is also an ob system. Since the focus of this paper is on formal objects, we are primarily interested in the "pure morphology" of ob systems. To give a simple example, consider a single atom a and a binary operation $\langle\ ,\ \rangle$. The obs are a, $\langle a, a\rangle$, $\langle\langle a, a\rangle, a\rangle$, $\langle\langle a, a\rangle, \langle a, a\rangle\rangle$, etc.

For an even simpler example take an atom 0 and a unary operation S. The resulting inductive class of formal objects, which consists of 0, $S0$, $SS0$, $SSS0$, and so on, can be employed as a basis for arithmetic. To

[4] For a comparison of Curry's notion of a formal system with others discussed in the literature see [Cur63, Sect. 2S].

this end, consider the formal system consisting of these obs, a two-place basic predicate $=$, the axiom $0 = 0$, and the deductive rule that if $x = y$ then $Sx = Sy$. Here x and y are unspecified obs, i.e., the rule is actually a *scheme* that gives rise to infinitely many rules (cf. p. 55). Peano's third axiom, for instance, which corresponds to the rule that if $Sx = Sy$ then $x = y$, is *admissible* in the sense that adjoining it to the system does not affect the class of elementary theorems [p. 256].[5] The rule is thus an *epitheorem* of the system, i.e., a provable statement about elementary statements.[6] Without going into details let us note that addition and multiplication can be introduced into the system by *definitional extension* [p. 107]. For a full account of this formal system as a basis for arithmetic, the reader is referred to [Sel75].

We close our brief exposition of formal systems by introducing the notion of a *representation*, which is essentially a structure preserving one-to-one correspondence between the formal objects of a formal system and certain objects "given from experience":[7]

> Any way of regarding the formal objects as specified objects given from experience will be called a *representation* of the system, provided the contensive objects retain the structure of the formal objects. [. . .] It means that there is a separate contensive object for each formal object, and [. . .] that the operations must be reflected in some way as modes of combination of the contensive objects. [. . .] In technical terms there must be a one-to-one correspondence, isomorphic with respect to the operations and modes of combination, between the formal objects [. . .] and the contensive objects of the representation. [p. 57]

For instance, the formal objects can be represented by their *names* that are introduced by the specification of the formal system; this is called the *autonymous* representation [p. 57]. So, every formal system has a syntactical representation. Another example is the *Gödel representation*, where each formal object is assigned its Gödel number [p. 58]. Curry emphasizes that it is "possible to present a system without having any specific representation in mind" [ibid]. He calls such a system *abstract*.

---

[5] Cf. also [Lor69].

[6] Curry uses the prefix 'epi-' instead of the more common 'meta-'.

[7] For the sake of completeness, we should also mention Curry's notion of an *interpretation* of a formal system, which resembles to some degree the standard concepts of model theory; cf. pp. 59f.

3 Logic and Language

Let us now look more closely at Curry's conception of logic and contrast it with Quine's views on logic and language. Since Curry tries to provide mathematical logic with a foundation by means of formal systems, it is not surprising that in his opinion his formalist approach does not hinge on logic at all:

> [...], from the standpoint of formalism [...] one can characterize a mathematical system objectively without presupposing anything which would be natural to call "logic".
>
> [Cur63, p. 18]

Presumably this does not include *philosophical logic*, which, according to Curry, is concerned with the "principles of valid reasoning" [*op cit*, p. 1]. For we may assume that he considers his reasoning valid too.

However, it is hard to draw a line between philosophical and *symbolic* logic if the latter is taken as concerned with the regimentation and formalization of the logical structure of ordinary language and the principles of valid reasoning. According to Quine, "[t]he effect of the regimentation is to reduce grammatical structure to logical structure"[8] and "to paraphrase a sentence of ordinary language into logical symbols is virtually to paraphrase it into a special part still of ordinary or semi-ordinary language".[9] Since one of the main purposes of regimentation is to resolve ambiguities in ordinary language and thereby to increase conceptual clarity, Curry as a scientist should welcome such a move – whether or not logical symbols are used instead of ordinary language expressions.

At least, Curry is quite aware of the importance of language in describing and communicating formal systems:

> The construction of a formal system has to be explained in a communicative language understood by both the speaker and the hearer. We call this language the *U-language* (the language being used). [...] It is well determined but not rigidly fixed; new locutions may be introduced in it by way of definition, old locutions may be made more precise, etc. Everything we do depends on the U-language; we can never transcend it; whatever we study we study by means of it. Of course, there is always vagueness inherent in the U-language; but we can, by skillful use, obtain any degree of precision by a process of successive approximation.
>
> [CF58, p. 25]

Although at first glance, Curry's goal of a sufficiently precise language seems to be in accordance with Quine's program of regimentation, there

---

[8] [Qui87, p. 158].
[9] [Qui60, p. 159].

is a big difference in attitude. For Quine, regimentation means reformulation in a restricted language that allows a direct symbolization within the language of quantificational logic, because he regards formulability in that language more or less as equivalent to full intelligibility.[10]

Curry, in contrast, does not give an explicit criterion of precision. For instance, he regards the notions of effective process and class or totality as sufficiently explicated in the form presented in Section 2.1 above. In particular, he does not address the question of ontological commitment, which for Quine is intimately connected to properly analyzing and regimenting the discourse in question.

Before we will take up this issue in more detail in Section 4.1, let us dwell a bit on the status of logic and language in Quine's overall picture of scientific inquiry. Quine thinks of "logic as the grammar of strictly scientific theory."[11] So, in a sense, language and logic are prerequisite to science.[12] But if language is regarded as essential for science – including the science of formal systems – and if the language in question, though ordinary, is regimented in certain ways, then it is fair to ask for a precise definition of that very language. At this point, a problem arises: we cannot devise a theory of regimentation that meets Quine's scientific standards without getting into an infinite regress.

The same problem is virulent when logical reasoning is at issue. In his *Methods of Logic* (as well as in other writings), Quine uses so-called *schematic letters* as a notational device for specifying logically valid sentence and inference schemata. Schematic letters, say 'p' and 'q' in '$p \& q \rightarrow p$', are to be treated as "placeholders" for expressions, here sentences, and not as variables ranging over the members of some universe. With Alex Orenstein we can question the status of these placeholders:

> We are told that schematic letters [. . .] are neither object language expressions nor metalinguistic variables. This is only a negative characterization and out of keeping with Quine's requirement for being precise. Worse still, the introduction of schemas involves positing additional types of expressions and additional rules determining their wellformdness.
>
> [Ore02, p. 116]

Quine might respond that schematic letters are a convenient but dispensable device. He could refer us to his method of *quasi-quotation*,

[10] Cf. e.g. [Qui60].

[11] [Qui01, p. 219].

[12] This observation has to be qualified insofar that Quine considers logic a part of his holistic *web of belief* and thus open to revision, at least in principle.

which allows to quantify over parts of expressions, and which is definable within a fully formalized theory of expressions, called *protosyntax*, as demonstrated in [Qui51, Chap. 7]. This response, however, can be criticized on two grounds: First, any protosyntactical theory already makes use of elementary logic. Second, defining concatenation in terms of writing and inscriptions, as done in [Qui51, p. 288], is insufficient for protosyntax, as Quine himself points out, for example, in [Qui69a, p. 42], where he proposes to employ finite sequences instead, whose definition in turn employs the natural numbers.[13] So, there can be no first theory of regimentation and logical inference as a scientific theory about the expressions of a natural or formalized language.

The alternative is to take up an ontogenetic point of view. Quine endorses the assumption that "the basic laws of logic [. . .] are internalized in childhood, in acquiring the use of the logical particles 'not', 'and', 'or', 'some', 'every'."[14] Learning elementary logic is thus on a par with learning to master your mother-tongue.[15]

To sum up, for Quine, elementary logic (the principles of valid reasoning) is an indispensable part of any serious science. Moreover, he regards regimentation into the (externalist) language of predicate logic (his canonical notation) as a strong requirement for full intelligibility. On the other hand, and Quine is everything but explicit on this point, elementary logic cannot be described in a way meeting these standards without giving rise to an infinite regress. This negative conclusion should not be too surprising since Quine repudiates first philosophy anyway. So Quine's attitude is probably better seen as normative rather than as foundational.

Curry, in contrast, does not address the logical structure underlying his presentation of formal systems. He seems to take "the principles of valid reasoning" as obvious.[16] But there is every reason to make logical structure and reasoning explicit in order to detect hidden assumptions and to avoid errors. In the words of Donald Davidson: "By prompting us to decide on the logical form of sentences [the program of formalizing a natural language] can reveal our basic ontological commitments, it can

[13] Noticeably, Quine [Qui46] also espouses the idea of reducing arithmetic to protosyntax.

[14] [Qui95a, p. 51].

[15] But notice that logic, according to Quine and contra Carnap, cannot be learned by convention, because conventionalism already presupposes language and logic; cf. [Qui76].

[16] There are some exceptions; see, for instance, the preliminary remarks about the logical connectives in the context of epitheoretic reasoning; cf. [Cur63, pp. 96f].

tell us where inferences are truly logical and where they are not, and it can reveal problems we had barely appreciated."[17]

4 Structure and Ontology

4.1 Ontological Commitment and Individuation

One of the hallmarks of Quine's philosophy is "his insistence upon being scrupulously clear and consistent about one's ontological commitments."[18] These commitments are manifest in (regimented) discourse:

> We can very easily involve ourselves in ontological commitments by saying, for example, that *there is something* (bound variable) [. . .] which is a prime number larger than a million. But this is, essentially, the *only* way we can involve ourselves in ontological commitments: by the use of bound variables. [Qui61, p. 12]

Hence the famous slogan "to be is to be the value of a bound variable." Notice that this slogan only indicates how to reveal ontological commitments and not whether they are acceptable in discourse, scientific or otherwise:

> We look to bound variables in connection with ontology not in order to know what there is, but in order to know what a given remark or doctrine, ours or someone else's, *says* there is; and this much is quite properly a problem involving language. [Qui61, pp. 15f]

For Quine, a necessary requirement for accepting an ontological commitment is to be able to formulate criteria as to whether any two of the postulated entities are identical or not; in short: no entity without identity. For "[w]e cannot know what something is without knowing how it can be marked off from other things. Identity is thus of a piece with ontology."[19] Intersubjective science is hardly possible if two scientists would not be able to make sure that they are talking about the same thing.

Let us reconsider the key notions underlying Curry's definition of formal systems under this perspective; cf. Section 2.1 above. First of all, he commits himself to the existence of *conceptual classes* as "totalities of elements defined in a strictly intuitive way." So, when are two conceptual classes X and Y identical? To say that they are identical if everybody has the intuition that they are should surely not count as an acceptable criterion. Happily, there is a rather straightforward alternative: X and Y are

[17] [Dav99, p. 715].
[18] [Ore02, p. 24].
[19] [Qui69a, p. 55].

identical if they have the same elements, that is, with '$x \in X$' for 'x is an element of X', if $\forall x(x \in X \leftrightarrow x \in Y)$. In other words, conceptual classes are identical if they are *coextensive*. Curry would presumably agree. The more pressing problem is to pin down the conceptual classes Curry commits himself to exist. His reference to a "strictly intuitive" way of definition is of minor use to anybody lacking Curry's intuitions. Moreover, he agrees that (naive) intuition gives rise to *Russell's paradox*.[20] Admittedly, Curry's primary interest is in *inductive* classes.

4.2 Inductive Constructions

Recall from Section 2.1 that an inductive class is generated from certain initial elements by certain modes of combination. To individuate classes, one needs to individuate their elements. So we need to state identity conditions for the elements of an inductive class. Let us confine our discussion to monotectonic classes. Then, by definition, two elements are identical if and only if they have the same construction. This leaves us with identity conditions for processes that consist in iterated applications of the modes of combination. We can take for granted that two modes of combination are identical if they take the same arguments to the same values; hence we can as well speak of *functions* or *functional relations* instead. But how to individuate processes of iterated applications of them? Certainly, they are not meant to be processes in space and time nor mental processes of a particular person. Apparently, we are hopelessly thrown upon Curry's appeal to intuition.[21]

From a Quinean perspective, we better dispense with processes altogether and explicate the notion of iteration in the first place. Let us assume for the moment the natural numbers as given. Then the *iterate* of a function or relation can be defined in terms of *finite sequences*.[22] Consider, for example, the monotectonic inductive class given by a class A of initial elements and a single binary function f such that $f(x, y) = f(u, v)$ only if $x = u$ and $y = v$. Let g be the function that takes a given class X to $X \cup f(X \times X)$. Then x is an element of the generated inductive class if and only if there is a natural number n such that $x \in g^n(A)$. As

[20] [Cur63, p. 4].

[21] It is worth mentioning that even Charles Parsons, who concedes intuition a certain role in grasping mathematical objects, is "inclined to deny that even very simple inductive conclusions are intuitive knowledge" [Par80, Sec. VIII].

[22] See [Qui69b, §14].

for the ontological commitments underlying this explication, a careful analysis shows that if we are only inclined to posit the generated formal objects and not the whole class of them, then the only postulates needed are the existence of the classes $\{0, 1, \ldots, n\}$ and *replacement* on them; see [Qui69b] for details.[23] In other words, by defining a formal object we need to refer to the class of objects it is "built of", which is a rather modest assumption if anything is.

Of course, if the question is how to individuate natural numbers we should not take them as already individuated. Curry does not share such scruples, for he freely utilizes numbers and finite sequences in his specification of inductive classes. Interestingly, when it comes to explaining his informal use of the natural numbers, Curry [Cur63, p. 42] refers to the following characterization:

> Any system of objects, no matter what, which is generated from a certain initial object by a certain unary operation in such a way that each newly generated object is distinct from all those previously formed and that the process can be continued indefinitely, will do as a set of natural numbers. [Cur63, p. 12]

But this is essentially a specification of a monotectonic inductive class, with the notions of process and generation as unexplicated as ever.

Quine [Qui69b, Chap. IV] offers a definition of the natural numbers along the following lines: Take 0 as anything you like, and take as S any function such that $S(x) \neq 0$, for every x, and $S(x) = S(y)$ only if $x = y$. Then x is a natural number if

$$\forall X(x \in X \mathbin{\&} \forall y(S(y) \in X \to y \in X) \to 0 \in X).$$

Quine's idea is that his "inverted" definition circumvents the need of infinite classes because the outer variable is required to range only over finite sets – in contrast to the classical definition of the closure specification by Frege, Dedekind, and Peano.[24] In order to make sure that there are enough finite sets, we can follow von Neumann and define 0 as $\{\}$ and $S(x)$ as $x \cup \{x\}$. Notice that assuming the existence of these sets involves genuine ontological commitments since we are not presupposing any axiomatic set theory. The existence of a natural number thus hinges on the existence of the class of all predecessors of that number.

[23] In particular, there is no need to postulate the existence of the union or the product of classes; for to say that $x \in g(X)$ is just a convenient way of saying that $x \in X \lor \exists y \exists z(y \in X \mathbin{\&} z \in X \mathbin{\&} x = f(y, z))$.

[24] See also the discussion in [GV98, pp. 321f]. An alternative approach within weak second-order logic, with separate variables for finite sets, is proposed by [FH95].

4.3 "The Ordered Pair as a Philosophical Paradigm"

Although the characterization of inductive constructions is an important issue for the science of formal systems, one can study the nature of formal objects also by looking at such elementary constructs as the *ordered pair*. In his presentation of formal systems, Curry does not mention ordered pairs at all. Quine, on the other hand, devotes a whole section of his *Word and Object* to the ordered pair.[25] To motivate that the notion of an ordered pair calls for explication, Quine cites the following characterization by Peirce:

> The Dyad is a mental Diagram consisting of two images of two objects, one existentially connected with one member of the pair, the other with the other; the one having attached to it, as representing it, a Symbol whose meaning is "First", and the other a Symbol whose meaning is "Second".[26]

Ordered pairs are in charge when binary relations are taken as classes (of ordered pairs). They are then typically used as values of variables of quantification and are thus to be treated as entities. As to the question what ordered pairs are, Quine points out that two ordered pairs $\langle x, y\rangle$ and $\langle u, v\rangle$ are identical if and only if $x = u$ and $y = v$. This identity condition is the only thing that matters when referring to ordered pairs. In *Word and Object*, Quine puts forward the slogan that "explication is elimination", which means to systematically choose already-recognized objects as ordered pairs subject to the restriction that they satisfy the identity condition. Since he aims at ontological economy, his preferred candidates are set theoretic constructs like Kuratowski's $\{x, \{x, y\}\}$. For him, "the question 'What is an ordered pair?' is dissolved by showing how we can dispense with ordered pairs in any problematic sense in favor of certain clearer notions."[27]

In later writings, Quine shifts emphasis more towards ontological relativity and indifference. The point is now not so much one of reduction or elimination but one of positing entities that satisfy such and such conditions. In order "to affirm something about a pair $\langle u, v\rangle$, and to do so without choosing any one of the various ways of constructing ordered pairs",[28] Quine considers to employ the technique of *Ramsey sentences*. A sentence about $\langle u, v\rangle$, say '$P\langle u, v\rangle$', is then replaced by

[25] [Qui60, §53], from which the present section borrows its title.
[26] cited after [Qui60, p. 257].
[27] [Qui60, p. 260].
[28] [Qui95a, p. 74]

$$\exists f(\forall x\forall y\forall z\forall w(fxy = fyw \leftrightarrow x = z \,\&\, y = w) \,\&\, P(fuv)).$$

As Quine observes, “Ramsey’s treatment [. . .] brings out indeterminacy of reference not by reinterpretation, but by waiving the choice of interpretation.” He furthermore observes that “each Ramsey sentence is a fresh existential quantification; consequently there is no assurance of sameness of objects from sentence to sentence”, which he sees as unproblematic in the case of formal objects because “they can be happily dismissed after each application and introduced anew for the next.” We should add, however, that the existential quantification has to take scope over the whole discourse of the application in question.

The ontological scrupulous might hesitate to posit functions in order to cope with ordered pairs.[29] A more modest solution could run as follows: Take the locution ‘the order pair of x and y’ as a definite description, i.e., if ‘$Fxyz$’ stands for ‘x is the ordered pair of y and z’, then ‘$\langle y, z\rangle$’ is short for ‘$\iota xFxyz$’. Now eliminate the definite description in favor of a uniqueness and an existence assumption. The exact treatment of the latter is open to various options. Russell, for instance, regards the existence assumption as part of the sentence under consideration, whereas Peano and Hilbert take it as a presupposition. Although these differences are of interest to our discussion, especially the scope considerations in the case of Russell’s contextual definition, lack of space prevents us from going into details.

It is tempting to suggest that there is nothing more to say about the nature of ordered pairs than to require existence and uniqueness and the condition that ordered pairs are identical only if their components are identical. This view seems to be in full accordance with Hilbert’s axiomatic method. The question ‘What are ordered pairs?’ is then answered by saying that ordered pairs are something we assume to exist and to have such and such identity conditions. Quine’s above argument that things become clearer if ordered pairs are identified with sets, say with Kuratowski’s representation, is not convincing. For what are sets? The only legitimate answer can be: entities we assume to exist with such and such identity conditions.

[29] Quine, in fact, speaks of an “unrealistic” assumption. What he presumably has in mind is that functions are usually defined in terms of ordered pairs (or triples, etc). However, though ordered pairs are technically useful to define relations and functions as sets of ordered pairs, one can do without them, as already done so in *Principia Mathematica*.

4.4 Structuralism

Structuralism is, as Charles Parsons puts it, "the view that reference to mathematical objects is always in the context of some background structure, and that the objects involved have no more to them than can be expressed in terms of the basic relations of the structure".[30] The slogan is that formal objects are nothing but "positions" in structures.

Quine is a confessing structuralist – not only with respect to *formal* objects. He holds the view of a global structuralism, which is closely connected to his doctrine of *ontological relativity* or the *inscrutability of reference*:

> [. . .] if we transform the range of objects of our science in any one-to-one fashion, by reinterpreting our terms and predicates as applying to the new objects instead of the old ones, the entire evidential support of our science will remain undisturbed.
> [. . .] there can be no evidence for one ontology as over against another, so long anyway as we can express a one-to-one correlation between them. Save the structure and you save all.
> [Qui92, p. 8]

For Quine, considerations of this sort "belong not to ontology but to the methodology of ontology, and thus to epistemology."[31]

Michael Resnik, who is also inclined to prefer an epistemic interpretation of structuralism, points out that one can adhere to a structuralist ontology in mathematics without committing oneself to ontological structuralism "all the way down", i.e., without saying that all objects are literally positions in structures.[32] However, Resnik is only prepared to accept an ontological reading of structuralism that posits "positions" as objects but not the structures themselves.

The version of structuralism favored by Stewart Shapiro, in contrast, treats structures as genuine objects. Shapiro maintains that "[Quine's] thesis of inscrutability blocks the final ratification of structuralism."[33] At the same time, Shapiro is well aware of the epistemic link between ontology and language:

> [. . .] grasping a structure and understanding the language of its theory amount to the same thing. There is no more to understanding a structure and having the ability to refer to its places than having an ability to use the language correctly. [Sha97, p. 137]

Quine could surely agree. But Shapiro seems to be after a true ontology which is independent of our epistemic grasp, whereas for Quine, "to

[30] [Par90, p. 303].
[31] [Qui81, p. 21].
[32] See [Res97, Sect. 12.8].
[33] [Sha97, p. 141].

ask what reality is *really* like [...] apart from human categories, is self-stultifying."[34]

Quine's dictum that explication is elimination gives his structuralism an *eliminative* connotation. Explication of the natural number structure, for instance, means for him to eliminate it in favor of any progression, preferably of sets:[35]

> I prefer to say with Benacerraf simply that there are no natural numbers, and there is no need of them, since whatever purposes we might have used them for can be served by any progression, and set theory affords progressions in generous supply. [Qui98, p. 403]

Here, we can argue as in the case of ordered pairs that aside from ontological economy there is no reason not to grant natural numbers (or other formal objects) the same ontological status as sets. One can nevertheless be scrupulous with respect to explicitly positing structures in addition to positions in structures, whereas the unscrupulous may employ the technique of Ramsey sentences as indicated in our discussion of ordered pairs. Shapiro in essence takes the latter route, via second order logic and implicit definitions.

After this brief overview of structuralist positions,[36] let us seek for traces of structuralism in Curry's formalist framework. According to the above characterization of structuralism by Parsons, we need to address the following two questions: does it make sense to refer to formal objects, i.e., to Curry's obs, without having a formal system in background, and is there more to a formal object than its relation to the other objects of the system?

The negative answer to the first question is immediate since any ob belongs to an inductive class defined by some ob system. As to the second question, recall that an ob is uniquely determined by its construction; in fact, "an ob can be identified with [...] a construction, objectified, if you will, by means of a tree diagram (or a normal construction sequence)."[37] An ob is thus fully characterized by two things: a certain mode of combination and a certain tuple of obs, where the ob in question is the result of applying the mode of combination to the given tuple. Under the reanalysis given in Section 4.2 this comes down to saying that an ob is fully

---

34 [Qui92, p. 9].

35 See also [Qui69a, pp. 44f].

36 It should be noticed that there are further variants of structuralism; see, for instance, Geoffrey Hellman's *modal* structuralism [Hel89].

37 [Cur63, p. 54].

characterized by a certain functional relation the ob in question bears to certain other obs of the system – and there is apparently nothing more to say of an ob.

We can conclude that the structuralist viewpoint is well suited for inductive classes and hence for ob systems.[38] Furthermore notice that Curry's characterization of the natural numbers cited above in Section 4.2 cannot deny a rather strong structuralist flavor.

At the close of Section 2.2, we saw that for Curry, formal systems can be *abstract* in that it is "possible to present a system without having any specific representation in mind." In spite of Curry's reluctance concerning explicit ontological commitments, it is tempting to read the foregoing statement to the effect that formal systems or, better, the structures determined by formal systems, are posited as genuine (abstract) objects.

5 Conclusion

In sum, Curry's position concerning the nature of formal objects appears to be compatible with the basic assumptions of structuralism – and thus in some respect with a Quinean viewpoint. We can also concede a considerable agreement between Curry and Quine about the importance of language in accessing formal objects.

There is a stark contrast between their attitudes towards the role of logic and its relation to language. While Curry draws a sharp line between mathematical and philosophical logic, which goes along with a distinction between different levels of language, Quine prefers "the fiction of an all-purpose scientific language",[39] with logic as its grammar. For Quine, ontological commitments are intimately connected to the logical syntax of language, whereas Curry does not address such questions at all.

Curry aims at a foundation of mathematical logic on the basis of formal systems. What makes his approach particularly interesting to the study of formal objects is his departure from Hilbert in that he does not restrict himself to syntactical systems. All in all, it might be worthwhile to continue the reanalysis of Curry's approach to formal objects under a Quinean perspective – as sketched in this essay – thereby retaining its constructivist appeal without falling back on the intuitively given.

[38] Cf. also the discussion in [Sha97, Chap. 6] on structuralist interpretations of constructivism.
[39] [Qui91, p. 243]

References

[CF58] Haskell B. Curry and Robert Feys. *Combinatory Logic*, volume 1. North-Holland, Amsterdam, 1958.

[Cur51] Haskell B. Curry. *Outlines of a Formalist Philosophy of Mathematics*. North-Holland, Amsterdam, 1951.

[Cur63] Haskell B. Curry. *Foundations of Mathematical Logic*. McGraw-Hill, New York, 1963.

[Dav99] Donald Davidson. Reply to Ernie Lepore. In Lewis Edwin Hahn, editor, *The Philosophy of Donald Davidson*, The Library of Living Philosophers, Volume XXVII, pages 715–717. Open Court, Chicago, IL, 1999.

[FH95] Solomon Feferman and Geoffrey Hellman. Predicative foundations of arithmetic. *Journal of Philosophical Logic*, 24:1–17, 1995.

[GV98] Alexander George and Daniel J. Velleman. Two conceptions of natural numbers. In Garth Dale and Gianluigi Oliveri, editors, *Truth in Mathematics*, pages 311–327, Oxford, 1998. Oxford University Press.

[Hel89] Geoffrey Hellman. *Mathematics without Numbers*. Oxford University Press, Oxford, 1989.

[Lor69] Paul Lorenzen. *Einführung in die operative Logik und Mathematik*. Springer, Berlin, 2nd edition, 1969.

[Ore02] Alex Orenstein. *W. V. Quine*. Acumen, Chesham, UK, 2002.

[Par80] Charles Parsons. Mathematical intuition. *Proceedings of the Aristotelian Society*, 80:145–168, 1979–1980.

[Par90] Charles Parsons. The structuralist view of mathematical objects. *Synthese*, 84:303–346, 1990.

[Qui46] W. V. Quine. Concatenation as a basis for arithmetic. In Quine [Qui95b], pages 70–82.

[Qui51] W. V. Quine. *Mathematical Logic*. Harvard University Press, Cambridge, MA, 2nd edition, 1951.

[Qui60] W. V. Quine. *Word and Object*. MIT Press, Cambridge, MA, 1960.

[Qui61] W. V. Quine. On what there is. In *From a Logical Point of View*, pages 1–19. Harvard University Press, Cambridge, MA, 2nd edition, 1961.

[Qui69a] W. V. Quine. Ontological relativity. In *Ontological Relativity and Other Essays*, pages 26–68. Columbia University Press, New York, 1969.

[Qui69b] W. V. Quine. *Set Theory and its Logic*. Harvard University Press, Cambridge, MA, 2nd edition, 1969.

[Qui76] W. V. Quine. Carnap on logical truth. In *The Ways of Paradox and Other Essays*, pages 107–132. Harvard University Press, Cambridge, MA, 2nd edition, 1976.

[Qui81] W. V. Quine. Things and their places in theories. In *Theories and Things*, pages 1–23. Harvard University Press, Cambridge, MA, 1981.

[Qui87] W. V. Quine. *Quiddities: An Intermittently Philosophical Dictionary*. Harvard University Press, Cambridge, MA, 1987.

[Qui91] W. V. Quine. Immanence and validity. In Quine [Qui95b], pages 242–250.

[Qui92] W. V. Quine. Structure and nature. *Journal of Philosophy*, 89(1):5–9, 1992.

[Qui95a] W. V. Quine. *From Stimulus to Science*. Harvard University Press, Cambridge, MA, 1995.

[Qui95b] W. V. Quine. *Selected Logic Papers*. Harvard University Press, Cambridge, MA, 2nd edition, 1995.

[Qui98] W. V. Quine. Reply to Charles Parsons. In Lewis Edwin Hahn and Paul Arthur Schilpp, editors, *The Philosophy of W. V. Quine*, The Library of Living Philosophers, Volume XVIII, pages 396–403. Open Court, Chicago, IL, 2nd edition, 1998.

[Qui01] W. V. Quine. Confessions of a confirmed extensionalist. In Juliet Floyd and Sanford Shieh, editors, *Future Pasts: The Analytic Tradition in Twentieth-Century Philosophy*, pages 215–221. Oxford University Press, Oxford, 2001.

[Res97] Michael D. Resnik. *Mathematics as a Science of Patterns*. Oxford University Press, Oxford, 1997.

[Sel75] Jonathan P. Seldin. Arithmetic as a study of formal systems. *Notre Dame Journal of Formal Logic*, 16(4):449–464, 1975.

[SH80] Jonathan P. Seldin and J. Roger Hindley. A short biography of Haskell B. Curry. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages vii–xi. Academic Press, London, 1980.

[Sha97] Stewart Shapiro. *Philosophy of Mathematics*. Oxford University Press, Oxford, 1997.